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AN ELEMENTARY TREATISE

ON

CURVE TRACING.

BY

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PREFACE.

I do not much like the idea of writing a preface, but I feel myself obliged to say a few words on the publication of what I have called a treatise, the term being very likely a misnomer.

Although my subject is Curve-tracing and not Curves, I am aware that some complete branches of this art are not alluded to at all.

The student might expect, in a treatise upon this subject, to find methods of drawing Polar Curves, Rolling Curves, Loci of Equations in Trilinear Coordinates, and Intrinsic Equations; he might also expect to find interesting Geometrical Loci discussed; these, and many other things immediately connected with the tracing of curves, have been deliberately omitted, for reasons which I consider good.

A treatise, if I had ventured upon it, at all comparable in exhaustive qualities with the excellent one of Salmon on Curves of Higher Orders, would have demanded, on the part of the student, far more extensive reading than I suppose him to possess; such a treatise would have required an advanced knowledge of Differential and Integral Calculus, of Higher Algebraical processes which do not appear in elementary treatises on Algebra, and of the science of projections, to understand

which involves a familiarity with Solid Geometry, beyond the standard to which I have supposed the student to have attained.

My readers must not be disappointed if they do not meet with an historical survey of the researches which have been made in old times on modes of generation and properties of particular curves, and in modern times on the singularities of curves; such a survey would have been irrelevant to the object which I have proposed to myself.

I acknowledge myself, nevertheless, indebted to many of those old mathematicians for ideas, and especially to Cramer, for many curves which I have employed in illustrating points on which I have been engaged.

In cutting off so many vital parts of a complete treatise, I have to shew that I do not fall to the ground by sawing on the wrong side the branch on which I am sitting; I shall therefore explain, in as few words as I can, the objects which I have had in view in my work as it stands.

In order to make any rapid progress, in after years, in all the difficult subjects to which mathematical analysis is applied, it is absolutely necessary that, by some means or other, a student should, as early as possible, make himself familiar with all the ordinary instruments of his trade, such as he handles when he studies Algebra, Trigonometry, and Algebraical Geometry; his tastes may carry him with greater impetus in one direction than another, but he should remember that

it is necessary to be strong all round, and even against the grain he should use efforts to avoid having weak points.

He must practise himself while he is young and his mind flexible, in all sorts of analytical processes and geometrical artifices. The solution of a great number of equations may be looked upon as one of the best exercises of one sort of faculties, and familiarity with the Binomial theorem and cognate subjects as essential, such as approximation to roots, expansion of a variety of functions in Algebra and Trigonometry, reversion of series, &c., accurate numerical calculations not being avoided.

I have reason to think that this kind of preliminary preparation for the study of the higher branches of mathematics has been much neglected in later years, and I am fortified in this opinion by observations made by Examiners of the greatest experience, who complain both of a want of power of work and of a want of individuality in the manner in which particular problems are attacked; this they attribute to defective early training and the omission of that practice which I have described as necessary.

Whether this practice has been neglected principally in consequence of the temptation to push forward to certain physical subjects, which have been recommended for the use of schools, I cannot say; but I have no doubt that the feeling of dignity acquired by entering upon the field of the Physical

Sciences has enticed many a student from a course which, if pursued, would have enabled him to do in a few weeks what it has taken him many months to puzzle over.

If there is time for a student not only to attend to the dry work of polishing, but also to make himself acquainted with a little Mechanics and Hydrostatics, he should by all means do so; but everyone who has examined in public schools knows how little time there is for the study of Mathematics, and how sensible the mathematical masters are of the insufficiency of this time. Looking, therefore, upon the total amount of energy as nearly constant, I should have no hesitation in reserving for some future time the study of the Physical Sciences, which will not eventually suffer; whereas, to attempt after a certain age to acquire ease in mathematical operations is like a grown man trying to learn the violin.

Having, then, a distinct feeling of the absolute necessity of developing skill and power—I will not add cunning—and, at the same time, being perfectly sensible in what dry places the poor spirit of a student has been condemned to wander in the performance of his duty, I have selected the subject of this work in order to relieve him in the dull work involved in his preparation for climbing heights, by taking him along a very pleasant path, on which he may exercise in an agreeable way all his mathematical limbs, and, if he keep his eyes open, may see a variety of things

which it will be useful to have observed when his real work begins.

For the subject, which I have chosen with this object in view, presents so many faces, pointing in directions towards which the mind of the intended mathematician has to radiate, that it would be difficult to find another which, with a very limited extent of reading, combines, to the same extent, so many valuable hints of methods of calculations to be employed hereafter, with so much pleasure in its present use.

For example, the subject of Graphical Calculation is coming more into use every day, and is applied with success to many difficult problems in Statics, Engineering, and Crystallography; hints of this the student will find in the practical solution of divers equations and in the determination of the number of their real roots, which are obtained by graphical methods with great facility.

Again, the methods of successive approximations which are employed in Optics and Astronomy are illustrated in the process of finding asymptotes and approximations to the forms of curves at a finite distance.

The comparison of large and small quantities of different orders of magnitude contains the staple of many of the most important applications of Mathematical Analysis; the Lunar and Planetary Theories depending almost entirely upon such considerations of relative magnitude.

The habit of looking towards an infinite distance, and discussing what takes place there, will render less startling a multitude of conceptions having in them a tendency to produce a feeling of vagueness, such, for instance, as the treatment of the mechanical effect of a couple as synonymous with that of an infinitely small force acting at an infinitely great distance.

As an important point, I would mention the tentative character of the inverse problem in which the form of a curve being given, its equation is investigated; the kind of uncertainty which will remain on the mind on account of defective estimation of magnitudes; and the necessity of a selection of what may appear the best of many possible solutions; all this will prepare the student for the disappointment which, having perhaps a wrong notion of what is meant by calling mathematics an exact science, he will feel in the conflict of theories by which it is attempted to reconcile the results of experiment in such subjects as Heat, Light, Electricity, and Molecular action generally; for an instance of this I may refer to the battle of philosophers about the direction of vibration of the ether in Plane Polarization.

The very uncertainty which exists in these subjects, the necessary balancing of evidence, and the difficulty of making up the mind as to what is to be believed, place such subjects, in the opinion of one at least of our greatest philosophers, among the best for the training of the intellect.

Looked upon as a special preparation for a special subject, I hope that my treatise may be considered useful in having given clear ideas, when the student enters upon the systematic treatment of the properties of curves; especially since the classification of curves according to degrees, and the subdivision of curves of the same degree into species is now being taken in hand by some eminent mathematicians.

With regard to the rejection of methods supplied by the Differential Calculus, I may observe that since the equations whose loci are investigated are rational equations, and never rise to a high degree, little would have been gained by the employment of such methods, since the Binomial Theorem is sufficient for all my purposes, and as ready in its application; independently of the consideration that I suppose myself to be instructing a student whose reading has been confined to very narrow limits.

As to the last chapter on Inverse Methods I trust that it will be looked upon as only a sketch. I have no doubt that the subject of it is capable of considerable perfection, and I shall be glad to have commenced, however defectively, so instructive a study.

To save the student trouble I may observe that I have used, as sufficiently near approximations in estimating their values, $\sqrt{3}=\frac{7}{4}$, $\sqrt{5}=\frac{9}{4}$, $\sqrt{7}=\frac{8}{3}$, $\sqrt{6}=\frac{5}{2}$, $\sqrt{10}=\frac{16}{5}$; and, with a view to the graphical solution of equations, I should ad-

vise him to practice himself in drawing a good parabola and in tracing readily the hyperbola from the equation $xy = ax + by + c$ for a variety of values of a, b, c.

In concluding this preface, or apology, I desire to say that I have read, with much advantage, some notes on Newton's enunciation of Lines of the Third Order by C. R. M. Talbot; and that I am indebted for some valuable hints to Mr. Clifford; but, especially, I must acknowledge myself in the highest degree indebted to two gentlemen, Mr. H. G. Seth Smith and Mr. G. L. Rives, of Trinity College, for their extreme kindness in guarding me against errors. The nature of the subject renders it extremely difficult to avoid mistakes; and, although very great pains have been taken to give correct drawings of the large number of curves which have been discussed, I am aware that there is much that is open to censure, and principally in parts for which my two friends are not at all answerable, many portions having been written when it was not possible to send them the proof sheets in time for revision.

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PLATE I.

directed to this subject, in a general way, without any attempt to do for curves of higher degrees what has been done so completely for curves of the second degree.

He will, for example, find himself better able to appreciate the illustrations which are used to make clear many theorems of the Differential Calculus, Theory of Equations, &c.; he will have definite ideas of the relative magnitudes of small quantities, and of infinitely great quantities; and especially he will become skilled in making correct approximations to the values of quantities, which cannot be found exactly, to any degree of accuracy which may be required.

And, at the same time, I hope that he will have much to interest him in the great variety which this application of Algebra to Geometry will open out to him.

INTRODUCTORY THEOREMS.

3. In order, as much as possible, to prevent interruption in the course of the work, I shall call attention to a few propositions with which some of my readers may not be familiar.

(1) If $\alpha, \beta, \ldots$ be n values, real or imaginary, called roots, which make $f(x)$, any rational and integral function of x, vanish, then $f(x)$ will be identically equal to $a(x-\alpha)(x-\beta)\ldots$ to n factors, supposing a to be the coefficient of the highest power of x in $f(x)$; consequently, the coefficient of x^{n-1} in $\frac{1}{a}f(x)$ is the sum of the n roots with their signs changed, that of x^{n-2} is the sum of the products taken two together, and so on.

(2) If $F(x, y)$ denote a homogeneous function of n dimensions of the form

$$ay^n + bxy^{n-1} + \ldots + gx^r y^{n-r} + \ldots + px^n,$$

and $m_1 m_2 \ldots$ be the n roots of

$$at^n + bt^{n-1} + \ldots + p = 0,$$

$F(x, y)$ will be identical with

$$a(y - m_1 x)(y - m_2 x)\ldots$$

and $F(x, y) = 0$ will represent n straight lines passing through the origin, real or imaginary.

If we write y^2 for y, $F(x, y^2)$ is a homogeneous function of x and y^2, and represents a series of parabolas, corresponding to the n values of $y^2 : x$.

As an example of the use of this, suppose the curve

$$x^3 + y^3 - 3axy = 0$$

to be cut by a straight line $lx + my = 1$ in the points P, Q, R.

Any relation obtained by combining these equations will be true for the points P, Q, R. One combination gives the homogenous equation

$$x^3 + y^3 - 3a(lx + my)xy = 0 \ldots\ldots\ldots\ldots (A),$$

which represents three straight lines through the origin, and consequently the lines OP, OQ, OR; and, if θ, ϕ, ψ be the angles of inclination of these lines to Ox,

$$\tan\theta \tan\phi \tan\psi = -1.$$

If we select l and m, so that $al = am = -1$,

(A) becomes $$(x+y)^3 = 0 \ldots\ldots\ldots\ldots\ldots\ldots (B),$$

the straight line PQR is

$$x + y + a = 0 \ldots\ldots\ldots\ldots\ldots\ldots (C),$$

(B) denotes that the line (C) meets the curve in three points at an infinite distance, being, as will be seen, an asymptote to the curve.

(3) If $f(x, y)$ and $\phi(x, y)$ be homogeneous functions of x and y, each of n dimensions,

and $$\frac{x}{l} = \frac{y}{m}, \text{ then } \frac{f(x, y)}{\phi(x, y)} = \frac{f(l, m)}{\phi(l, m)}.$$

For, let $$f(x, y) \equiv ax^n + bx^{n-1}y + \ldots + ky^n,$$

$$\phi(x, y) \equiv \alpha x^n + \beta x^{n-1}y + \ldots + \kappa y^n,$$

and $$\frac{x}{l} = \frac{y}{m} = r;$$

PLATE I.

therefore $$f(x, y) = (al^n + bl^{n-1}m + \ldots + km^n)\, r^n,$$

and $$\phi(x, y) = (\alpha l^n + \beta l^{n-1}m + \ldots + \kappa m^n)\, r^n;$$

whence $$\frac{f(x, y)}{\phi(x, y)} = \frac{al^n + \ldots + km^n}{\alpha l^n + \ldots + \kappa m^n} = \frac{f(l, m)}{\phi(l, m)}.$$

(4) In transforming the origin of coordinates from one point to another, the axes remaining parallel to their former position, it is required to substitute for x and y in an equation the values $x + \xi$, $y + \eta$; since the equations with which we have to deal are never of a very high degree, this can as easily be done, in particular cases, by common algebraical methods as by any other; but, for the sake of general statements, it is well to notice the law of formation of the coefficients in the expansion of $f(x + \xi)$, where $f(x)$ is a rational integral function of x.

Suppose gx^r to be one of the terms of $f(x)$ the coefficients of ξ, $\frac{1}{\lfloor 2} \xi^2$, $\frac{1}{\lfloor 3} \xi^3$ ... in $g(x + \xi)^r$ are, by the Binomial theorem, grx^{r-1}, $gr(r-1)x^{r-2}$, $gr(r-1)(r-2)x^{r-3}$... and similarly for each term of $f(x)$; hence, if $f'(x)$ be the coefficient of ξ, $f'(x)$ is obtained from $f(x)$ by multiplying each term by the index of the power of x in it, and then diminishing the index by unity. The coefficient of $\frac{1}{2}\xi^2$ is obtained from $f'(x)$ in the same way as $f'(x)$ from $f(x)$, and is written $f''(x)$, and so on, whence

$$f(x + \xi) = f(x) + f'(x)\,\xi + \tfrac{1}{2} f''(x)\,\xi^2 + \ldots$$

As an example, suppose we want to find the form of the curve

$$y = x - x^3,$$

in the neighbourhood of the point (1, 0), for x write $1 + \xi$, ξ being the abscissa with a new origin (1, 0); therefore

$$y = -2\xi - 3\xi^2 - \xi^3,$$

$y = -2\xi$ is very nearly true at the point, if ξ be made very small, try for instance $\frac{1}{20}$ or $\frac{1}{100}$, we have, as a still nearer approximation,

$$y = -2\xi - 3\xi^2;$$

PLATE I.

therefore y is a little less than 2ξ, whether ξ be positive or negative in the neighbourhood of the point $(1, 0)$.

The shape is therefore near this point like the figure. Fig. 1.

DEFINITIONS.

4. The following definitions are required:

The *equation of a curve* is an indeterminate equation between x and y, such that no point in the curve can be found whose coordinates do not satisfy the equation.

The *locus of an equation* is the assemblage of *all* the points whose coordinates satisfy the proposed equation.

A point of inflexion is a point at which the tangent to the curve cuts the curve, so that the curve bends as in the figure. Fig. 2.

A multiple point is a point in a curve through which more than one distinct branch passes; a multiple point may be double, triple, &c. as in the figures. Fig. 3, 4.

A point of osculation is a multiple point through which two branches pass which have a common tangent at that point. Fig. 5.

A cusp is a point at which two branches of a curve touch, but through which they do not pass.

A ceratoid cusp is one in which the branches lie on opposite sides of the common tangent. Fig. 6.

A ramphoid cusp is one in which they lie on the same side. Fig. 7.

A conjugate point is an isolated point, whose coordinates satisfy the equation, but through which no branches pass, such as a circle or ellipse would become if the diameter or axes were made indefinitely small.

All these points are called *Singular Points.*

An *asymptote* of a curve is a line towards which the curve finally approaches, as we recede from the origin to an infinite distance, and from which the distance may be made less than any assignable quantity.

An *asymptote* may be either curvilinear or rectilinear.

When a rectilinear asymptote is meant, this condition of indefinite approach is supposed to hold in every case.

PLATE I.

A proper curvilinear asymptote satisfies the same condition, but it is usual, for want of a better term, to call any curve an asymptotic curve which serves as a guide to the direction of the flexure in an infinite branch.

TRACING BY POINTS.

5. The most rudimentary way of tracing curves is to map down a number of points whose coordinates satisfy the equation of a curve, taken in some order, and so large that no branch may escape; if there be a doubt how to fill up any of the intervening spaces, more points must be interpolated.

(1) Take the curve

$$y = x(x^2 - 1),$$

$$\begin{array}{lccccccccc} x = & 0, & \tfrac{1}{4}, & \tfrac{1}{2}, & \tfrac{3}{4}, & 1, & \tfrac{5}{4}, & \tfrac{3}{2}, & \tfrac{7}{4}, & 2, \\ 64y = & 0, & -15, & -24, & -21, & 0, & 45, & 120, & 231, & 384. \end{array}$$

If the sign of x be changed, that of y is changed, the magnitudes being unaltered.

Fig. 8. If these points be mapped down as in the figure, a rough picture of the curve can be drawn without investigating the exact angles at which the curve cuts the axes, or the exact points at which the tangent to the curve is parallel to the axis of x, which we shall see is somewhere between $x = \frac{1}{2}$ and $\frac{3}{4}$, try for instance $\frac{5}{8}$.

(2) Again, take the curve $x^4 - 3axy^2 + 2ay^3 = 0$. Although we cannot solve the equation with regard to x or y, we can obtain a number of points by assuming $y = zx$, whence $x = (3 - 2z)z^2a$,

$$\begin{array}{lccccccccc} z = & 0, & \tfrac{1}{4}, & \tfrac{1}{2}, & \tfrac{3}{4}, & 1, & \tfrac{5}{4}, & \tfrac{3}{2}, & \tfrac{7}{4}, & 2, \\ 32x = & 0, & 5a, & 16a, & 27a, & 32a, & 25a, & 0, & -49a, & -128a, \\ 32y = & 0, & \dfrac{5a}{4}, & 8a, & \dfrac{81a}{4}, & 32a, & \dfrac{125a}{4}, & 0, & -\dfrac{343a}{4}, & -256a, \end{array}$$

$$\begin{array}{lcccccc} z = & \tfrac{5}{2}, & 3, & -\tfrac{1}{4}, & -\tfrac{1}{2}, & -1, & -2, \\ 32x = & -400a, & -864a, & 7a, & 32a, & 160a, & 896a, \\ 32y = & -1000a, & -2592a, & -\dfrac{7a}{4}, & -16a, & -160a, & -1792a. \end{array}$$

The points are mapped in the figure as far as space will allow. PLATE I. Fig. 9.

To obtain a dubious part of the curve near the origin, it would be sufficient to interpolate two more values in the neighbourhood of $z = \frac{3}{2}$,

$$z = \tfrac{3}{2} - \tfrac{1}{8} = \tfrac{11}{8},\ x = \tfrac{121}{256}a,\ y = \tfrac{11}{8}x,$$

$$z = \tfrac{3}{2} + \tfrac{1}{8} = \tfrac{13}{8},\ x = -\tfrac{169}{256}a,\ y = \tfrac{13}{8}x,$$

and notice that z which is $\tan POx$, diminishes as x changes from $-\frac{169}{256}a$ to $\frac{121}{256}a$.

In this manner it would be possible to give a very exact representation of any locus of an equation, within the limits of the paper, if we did not care for the trouble of interpolating values when the direction of the curve was at all doubtful.

The main object of a work on tracing curves must be to point out a variety of considerations which will relieve the student from a great deal of this labour, and enable him to indicate generally the peculiarities of a curve, its changes of direction and curvature, the interlacing of its branches, &c., while he can always for any particular purpose have recourse to more exact determination of special parts of the curve.

SYMMETRY OF A CURVE.

6. One of the first considerations is the symmetry of a curve with respect to certain lines or points, by means of which the labour is reduced one-half at once.

The principal kinds of symmetry arising from the form of the equation are as follows:

(1) If the rationalised equation of the curve involve only even powers of y, as

$$y^4 - b^2y^2 + ay^2x - cx^3 = 0,$$

the curve is symmetrical with respect to the axis of x, for if (x, y) be a point in the curve, $(x, -y)$ will also be in the

PLATE I. curve. The figure will be the same as if a plane mirror were placed perpendicular to the paper on the axis of x.

If the equation involve only even powers of x, there is symmetry with respect to Oy.

(2) If the equation be not altered when $-x$ and $-y$ are written for x and y, as when

$$x^4 - a^2xy + b^2y^2 = 0,$$

Fig. 10. this shews that, when P is a point on the curve, and PO is joined, and produced to P', making $P'O = PO$, P' is also a point on the curve. In other words, O is a centre of the curve, and the curve is symmetrical in opposite quadrants; so that, if yOx contained any part of the curve, and the figure were turned in its own plane through two right angles, it would overlie another portion of itself as it was in the original position.

(3) If the equation be unaltered when x and y are interchanged, as when

$$x^5 - 2a^3x^2 + 5a^3xy - 2a^3y^2 + y^5 = 0.$$

For every point $P(x, y)$ there is a point $Q(y, x)$, so that the curve is symmetrical with respect to a line bisecting the
Fig. 11. angle yOx, as in the figure.

Similarly, if $(-y, -x)$ may be substituted for (x, y) without altering the equation, as in

$$x^3 - 3axy - y^3 = 0,$$

this would shew symmetry with respect to a line bisecting the angle $x'Oy$.

(4) There are other kinds of symmetry, but it is scarcely worth the trouble of looking out for them except as tests of the tracing. Thus, $-x$ for y and y for x, may not alter the equation, as in

$$x^4 + a^2xy - y^4 = 0.$$

Such a curve would apparently have received no change
Fig. 12. if it were turned through a right angle in its plane.

I.

Employ the method given in Art. 3, to prove the following:

(1) If a tangent to a circle, whose equation is

$$x^2+y^2=\frac{a^2b^2}{a^2+b^2},$$

meet an ellipse, whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, in P and Q, prove that PQ subtends a right angle at the centre of the ellipse.

(2) The equation of a hyperbola is

$$xy-2x-3y+5=0.$$

find the two chords, through $(0, 2)$ which subtend a right angle at the origin.

Also to construct the loci of the following equations:

(3) $y^4-3axy^2+2a^2x^2=0$.

(4) $x^4-a^2x^2+2a^2xy-a^2y^2=0$.

Trace the curves whose equations are given below, by means of particular points.

(5) $(x-y)^2=ax$.

(6) $xy=c^2$.

(7) $xy-2x+y=0$.

(8) $y=x-2x^2+x^3$.

(9) $x^3+y^3-3axy=0$.

(10) $y^3=x^2(x-a)$.

CHAPTER II.

ORDERS OF SMALL QUANTITIES. FORMS OF PARABOLIC CURVES NEAR THE ORIGIN. CUSPS. TANGENTS TO CURVES. CURVATURE.

PLATE I.

7. If the student has taken the trouble to trace one or two curves by points, he will at once appreciate the advantage which it would be to him to have at every point the direction of the curve and that of its bend, as well as the point itself; for he would obtain a much more accurate representation of the curve with far fewer points.

The next thing I shall endeavour to make clear is the way to determine the shape of a curve in the neighbourhood of a point through which it passes; and since, by transformation, every point at a finite distance can be made a new origin of coordinates, it will be sufficient to discuss the form of the curve in the neighbourhood of the origin in cases in which it passes through it.

The form of the curve at points at an infinite distance will be discussed hereafter.

ORDERS OF SMALL QUANTITIES.

8. In handling this subject we are obliged to distinguish accurately such expressions as small, very small, infinitely small, vanishing, ultimately vanishing, large, infinitely large quantities, and to speak of things being equal, nearly equal, and so on; a few words may be useful to those who have not been in the way of dealing with variable quantities, which, although they vanish simultaneously, do not tend necessarily to equality though they both be zero.

PLATE I.

9. Finite quantities are said to be exactly equal when their difference is nothing.

Finite variable quantities become equal when their difference vanishes.

But it is evident that this is not a proper definition of equality of magnitudes which are themselves vanishing quantities. For, $2x$ is as truly double of x, when x is indefinitely small, as two inches are double of one, and yet the difference x vanishes. Hence the necessity of another definition.

Variable quantities are said to become equal, when their difference vanishes compared with either of them.

We may add that quantities are nearly equal, when the ratio of their difference to either of them is small.

10. Now, with regard to great and small things, it must be remembered that no quantity is absolutely small or great, but only so with reference to some unit either expressed or implied in the nature of the subject.

In measuring a degree, four yards would be a very large error, while in sweeping for the lost end of the Atlantic cable a couple of hundred yards was thought a small error in the supposed position.

In ancient times eight minutes was a small error in an astronomical observation; in modern times eight seconds is enormous.

Thus, when a unit has been determined in any subject in which a calculation is to be made, we must ask what is considered small—is it to be a tenth or a hundredth of the unit, or which?

Just as the unit of measurement was arbitrary, so the standard of smallness must be selected before the idea of what is small can be made definite for purposes of calculation.

Again, something more definite must be determined upon than the degrees of smallness or largeness expressed by the vague terms, very small or very large, extremely small or minute.

PLATE I.

For this purpose, having fixed upon what we determine to consider small, we introduce the terms small quantities of the second, third, &c. orders; the small quantities of the second order having the same ratio to those of the first order as those standard small quantities have to the unit employed.

Thus, in the Lunar Theory, fractions varying between $\frac{1}{12}$ and $\frac{1}{20}$ are called small compared with unity, $\frac{1}{12^2}$ a small fraction of the second order and so on, and complete calculations of the motion of the moon have been made to the sixth or seventh orders of small quantities.

If α be the fraction of the unit which is taken as the standard of smallness, and the exact value of u in terms of α be

$$a + b\alpha + c\alpha^2 + \ldots$$

to any number of terms, suppose that k is the greatest of the magnitudes b, c, ..., then $b\alpha + c\alpha^2 + \ldots < \frac{k\alpha}{1-\alpha}$, thus a differs from u by a quantity of the first order, and a is called the first approximation; similarly, $a + b\alpha$ is called the second approximation, &c.; these approximations imply that $k\alpha : a$ is of the order α, which circumstance must guide us in the selection of the standard.

It follows that whatever be the coefficients, still supposed finite, α can always be taken so small that a, $a + b\alpha$, ... are first, second, &c. approximations to the value of u, to any degree of accuracy.

11. When a variable quantity is supposed capable of being diminished until it becomes less than any assignable quantity, it is called a vanishing quantity, and if it be taken as the standard, finite multiples of its square, cube, &c. are called vanishing quantities of the second, third, &c. orders; so that, if u_1, u_2, u_3... be vanishing quantities of the first, second, third, &c. orders, the ratio $u_{n+1} : u_n$ is of the first order, $u_3 : u_1$ of the second, and so on; and if $v : u$ be a vanishing quantity, v is said to be of a higher order than u.

PLATE I.

FORMS OF PARABOLIC CURVES NEAR THE ORIGIN.

12. A graphic notion of the closeness of the approximations mentioned above will be obtained by tracing the curves

$$y=x^2,\ y=x^3,\ y=x^4,\ \ldots$$

from $x=0$ to $x=1$, using the same coordinate axes for all the curves, so that the relative magnitudes of the ordinates of the different curves, corresponding to any small value of x which we choose, may be seen, if x be not chosen too small, and conceived from the general run of the curves, if x be chosen extremely small. And the ordinates of these will be magnitudes of the second, third, ... orders, the standard being the chosen value of x.

In the figure, $OA=AB$ is taken as the unit, Fig. 18.

$$Oa=\tfrac{1}{2},\ Ob=\tfrac{1}{4},\ Oc=\tfrac{1}{8}.$$

The ordinates for the lines $y=x,\ x^2,\ x^3\ldots$

corresponding to a, are $a\alpha_1,\ a\alpha_2,\ a\alpha_3,\ a\alpha_4\ldots$

.... b, are $b\beta_1,\ b\beta_2,\ b\beta_3,\ \ldots$

.... c, are $c\gamma_1,\ c\gamma_2\ldots$

Each of the ordinates $a\alpha$ is $\frac{1}{2}$ of the preceding, each of $b\beta$ is $\frac{1}{4}$th of the preceding, each of $c\gamma$ is $\frac{1}{8}$th of the preceding, and the ordinates, corresponding to $Od=\frac{3}{4}$, have been constructed in order to guide to the general forms of the curves, five of which are placed in the figure.

We observe that, with a value of $x=\frac{1}{2}$, the ordinate of $y=x^6$ is very small, while, for $x=\frac{1}{4}$, we can distinguish neither the curve $y=x^5$ nor $y=x^6$, while for $x=\frac{1}{8}$, $y=x^4$ is not distinguishable from the corresponding point in the line of abscissæ.

If we take an abscissa $Og=\frac{1}{4}Oc$ or $\frac{1}{32}$, a length which is sufficiently visible, all the curves are coincident as far as we can see by our diagram.

Some idea can thus be formed of the nature of the approximations to $a+bx+cx^2+\ldots$ when x is excessively small.

PLATE I. 13. I have traced these curves by setting off particular points within the limits proposed, because in cases of this simplicity, the method sufficiently shows that there are no sinuosities, and it is easy to follow the form in the mind's eye up to any value of x however small.

Fig. 13. It is seen with respect to these curves that if we traced them for negative values of x, those with odd powers will occupy the opposite quadrant $x'Oy'$, as $O\beta_3'$, $O\alpha_3'$, and those with even powers will be symmetrical with respect to the axis of y, as $O\beta_2'$, $O\alpha_4'$.

If we trace beyond $x=1$, the ordinate of $y=x^4$ becomes greater than that of $y=x^3$, and they continue to diverge very rapidly from one another as x increases.

It may also be seen what the form of such a curve as $y^3=x^7$ would be, for it is intermediate between $y=x^2$ and $y=x^3$, and is symmetrical in opposite quadrants, since for every point x, y there is a point $-x$, $-y$.

It is not sufficient to consider the curve $y^2=x^3$ as lying between $y=x$ and $y=x^2$, we must also observe that all the curves which we have drawn touch the axis of x, because y is of a higher order than x, so that the curve $y^2=x^3$ touches the axis of x and is symmetrical with respect to that axis, also no part lies on the negative side of Oy, since x negative gives impossible values of y.

14. The importance of knowing the shapes of curves whose equations are of the form

$$y^m = cx^n$$

in the neighbourhood of the origin, will be seen hereafter; the student should exercise himself in drawing the forms for different values of m and n, as part of the machinery for tracing curves with facility; and also, in order to realise distinctly what is the relation among quantities of different orders of magnitude, two curves should be drawn with the same coordinate axes, and placed in their proper positions with reference to their degrees of closeness to the axis which they touch.

PLATE I.

Curves, whose equations are of the form $y^m = cx^n$, in which m and n are unequal, are called *parabolic curves.*

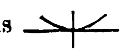
If m and n are both even, the locus of the equation is two distinct parabolas.

The arguments by which the forms in the neighbourhood of the origin are determined will pass through the mind very rapidly in the following form, with very little practice:

Take the case in which m is odd, and n is even, and less than m.

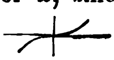
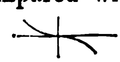
$-x$ written for x does not alter the equation; therefore the curve is symmetrical with respect to Oy.

x is small compared with y, therefore the curve touches the axis of y at O, hence the shape is the ceratoid cusp, as in the figure  .

If n be greater than m, the curve touches the axis of x at O, and the shape is  .

Take the case in which m and n are both odd integers.

$-x$ *and* $-y$ written for x and y does not alter the equation, hence the curve is symmetrical in opposite quadrants, or the origin is a centre.

If n be greater than m, the curve touches the axis of x, since y is small compared with x, and the shape is  if c be positive,  if c be negative.

15. If we now consider the fact that a quantity

$$ax^n + bx^r + cx^s + \ldots,$$

in which $n < r < s \ldots$, differs from any of the quantities

$$ax^n,\quad ax^n + bx^r,\quad ax^n + bx^r + cx^s \ldots$$

by a difference, which ultimately vanishes compared with itself, when x is indefinitely diminished, *or* that the ratio of the difference to itself may be made as small as we please by diminishing x, we shall be able to assign the direction taken by a curve in the neighbourhood of any particular point, by transferring the origin to that point.

PLATE I.

Thus, if the equation of the curve be $y^3 = x^4$, to find the tangent at the point (1, 1).

Let
$$x = 1 + \xi,\ \ y = 1 + \eta,$$
$$(1 + \eta)^3 = (1 + \xi)^4,$$
$$3\eta = 4\xi.$$

The tangent makes an angle $\tan^{-1}\frac{4}{3}$ with Ox.

Fig. 13. In the different curves, drawn in the figure, it can in this way be shewn that the tangents to the curves at B will, if produced, meet the axis of x at distances from A equal to $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$ respectively.

16. The equation of the tangent is thus easily found at any point, but the next approximation gives the direction in which the curve bends from the tangent.

Thus, if the equation of the curve be
$$y = x + x^3,$$
$y = x$ is the tangent at the origin.

Fig. 14. Draw MPQ, $M'P'Q'$ ordinates to the tangent at O, and the curve at small distances from O on opposite side, then
$$MP = \ \ x, \quad PQ = x^3,$$
$$M'P' = -x, \ \ M'Q' = -x - x^3.$$

The curve lies above on the positive side of O and below on the negative side.

The point O is a point of inflexion.

CUSPS.

17. The following examples will shew how certain peculiar forms of curves, such as cusps and points of inflexion, arise from fractional indices occurring in the equations of the curves, when y is expressed explicitly in terms of x, or *vice versâ*.

(1) $$y = 1 + x + 2(x-1)^2 + 3(x-1)^{\frac{5}{2}},$$
near the point (1, 2) let $y = 2 + \eta$, $x = 1 + \xi$,
$$\eta = \xi + 2\xi^2 + 3\xi^{\frac{5}{2}}.$$

PLATE I.

The first approximation gives $\eta = \xi$, so that the curve coincides with the line $Q'PQ$ nearly, which is therefore the tangent at P, if $OM = 1$, $MP = 2$, $\angle QP\xi = 45°$. Fig. 15.

The second approximation is $\eta = \xi + 2\xi^2$, therefore $2\xi^2$ being added to the ordinates of $Q'PQ$, the curve more nearly coincides with the dotted line $R'PR$, where $RQ = 2\xi^2$, which is a parabola, whose diameter through P is $MP\eta$.

The term added in the next approximation $3\xi^{\frac{5}{2}}$ is impossible when ξ is negative, and has two values equal and of opposite signs when ξ is positive, thus the form is SPS', where $RS = RS' = 3\xi^{\frac{5}{2}}$ is small compared with RQ, if ξ be taken very small. Hence, at P there is a *ramphoid cusp*.

(2) If the fractional index had been $\frac{7}{3}$, the form would have been SPS', in the next figure, since $3\xi^{\frac{7}{3}}$ only changes sign when ξ becomes negative. Fig. 16.

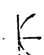
(3) If the fractional index be $\frac{3}{2}$, the terms must be re-arranged in the order

$$\eta = \xi + 3\xi^{\frac{3}{2}} + 2\xi^2.$$

The second approximation would give the form RPR', and the third would not be perceptibly different, since RS, $R'S'$ are small compared with QR or QR', and the only difference is, that the branch PS is further from PQ than PR, and PS' nearer than PR'; there is therefore at P a *ceratoid cusp*. Fig. 17.

(4) If $\frac{1}{2}$ be the fractional index,

$$\eta = 3\xi^{\frac{1}{2}} + \xi + 2\xi^2,$$

the curve is nearly the form of $\eta = 3\xi^{\frac{1}{2}}$, or $9\xi = \eta^2$. 

(5) If $\frac{2}{3}$ be the fractional index,

$$\eta = 3\xi^{\frac{2}{3}} + \xi + 2\xi^2,$$

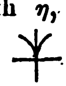
then $\eta = 3\xi^{\frac{2}{3}}$ nearly, ξ is small compared with η, and η remains the same if ξ changes its sign, hence  is the shape of the curve.

PLATE I.

TANGENTS.

17. At this point I think that the student, who has not read Newton's Lemmas, should be introduced to the notion of limits, as applied to the theory of tangents and the curvature of curves.

Two variable magnitudes whose variations depend upon that of a quantity which is supposed to diminish until it vanishes, are said to be *ultimately* equal, when the ratio of their difference to either of them vanishes, as the quantity upon which they depend vanishes.

The *limit* of a variable magnitude is that fixed quantity to which it is *ultimately* equal, when the variable on which it depends vanishes.

This includes the case of a variable, upon which the variation of the magnitude depends, increasing until it becomes infinite, because the reciprocal of such a variable diminishes until it vanishes, and the magnitude may equally be considered to depend on the reciprocal as a variable.

18. If OP be any curve passing through the origin of coordinates, OT the tangent at O, x, y the coordinates of

Fig. 18. any point P, $\tan POM = \frac{y}{x}$.

As OP diminishes, and ultimately vanishes, the angle POT ultimately vanishes, and $\tan TOM = \tan POM$ ultimately,

$=$ limit of $\frac{y}{x}$, when x and y vanish simultaneously.

The gradual diminution and ultimate evanescence of the angle POT may be seen by comparison with the curve

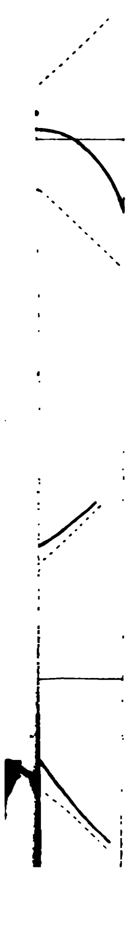
Fig. 13. $O\alpha_2 B$ in Art. 12, in which, if $O\alpha_2$, $O\beta_2$, $O\gamma_2$ be joined, the angles $\alpha_2 OA$, $\beta_2 OA$, $\gamma_2 OA$ exhibit the continual diminution of the angle between the chord and tangent.

Thus, if $y = \sin x$, since $\frac{\sin x}{x} = 1$ ultimately, when the unit is the unit of circular measure, the tangent at the origin is inclined at 45° to the axis of x.

g. 2

Fig.

ig II



2.

PLATE II.

Similarly, the tangent of the angle, which the tangent at any point (a, b) of a curve makes with Ox, is the limit of $\frac{y-b}{x-a}$, when $x-a$ and $y-b$ vanish, which they do simultaneously.

CURVATURE.

19. The only curve whose curvature is the same at every point is the circle, and the smaller the circle the greater is its curvature. Hence, the reciprocal of the diameter is taken as the measure of the curvature of a circle.

Suppose two circles have a common tangent AQ at A, diameters AB, Ab, QPp being perpendicular to the tangent AQ, meeting the circles in P, p, and PM, pm perpendicular to AB, then $AM.MB = PM^2 = pm^2 = Am.mb$; therefore $PQ.MB = pQ.mb$; or $PQ : pQ :: mb : MB$. Fig. 1.

If AQ be made to diminish indefinitely, PQ and pQ measure the deflection from the tangent, and the limit of the ratio $PQ : pQ$ is $Ab : AB$, or is equal to the ratio of the curvatures of the circles.

20. Any two curves AP, Ap, which have a common tangent at A, have the same curvature if $PQ = pQ$ ultimately.

If, therefore, a circle be drawn touching a curve at any point A, and its magnitude be such that $PQ = pQ$ ultimately, the circle has the same curvature as the curve at that point, and the curvature of that circle is the measure of the curvature of the curve at that point; the circle is called *the circle of curvature*, and its diameter *the diameter of curvature*.

21. No arc of any other circle can be drawn which lies between the curve and the circle determined above, for if possible, let an arc lie between them, cutting pPQ in p', then Qp' is intermediate in magnitude to QP and Qp which are ultimately equal, hence Qp' is ultimately equal to Qp, and the diameters of the two circles, being the limits of $\frac{AQ^2}{p'Q}$ and $\frac{AQ^2}{pQ}$, are equal. Fig. 2.

PLATE II.

22. Since, for a circle, $\frac{AQ^2}{pQ}$ is finite when the circle is of finite radius, it follows that when, at any point of a curve, $\frac{AQ^2}{PQ}$ ultimately vanishes, or becomes indefinitely great, there is no finite circle which has the same curvature.

A curve has finite curvature at A, if $\frac{AQ^2}{PQ}$ be ultimately finite, when AQ vanishes.

23. The diameter of curvature of a curve at A is the limit of either $\frac{AQ^2}{PQ}$ or $\frac{AP^2}{PQ}$, since $AP^2 = AQ^2 + PQ^2 = AQ^2$ ultimately.

II.

Find the forms, near the origin, of the curves whose equations are

(1) $a^3y^2 = x^5$. (2) $y^4 = ax^3$.
(3) $y^7 = a^2x^5$. (4) $a^2x^3 = y^5$
(5) $y = x + x^4$. (6) $y^2 = x^2 + x^3$.
(7) Find the tangent to the curve (2) at the point (a, a).

Find the form and position of the cusps in the following curves:

(8) $(y - x)^2 = x^3$ at the origin.

(9) $y - 2 = x^2 + x^{\frac{7}{4}}$ at the point (0, 2).

(10) $(2y + x + 1)^2 = 4(1 - x)$ at the point $(1, -1)$.

(11) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$ at the points where it meets the axes.

(12) $a^4y^2 = (x^2 - a^2)^3$ at the points $(\pm a, 0)$.

Find the diameters of curvature at the origin for the curves

(13) $y^2 = 4ax$. (14) $y^2 = 2mx + nx^2$.
(15) $ay(x - a) = (x - 2a)x^2$.

CHAPTER III.

FORMS OF PARABOLIC CURVES AT AN INFINITE DISTANCE. EXAMPLES OF TRACING CURVES. TRIGONOMETRICAL CURVES. ILLUSTRATIONS OF THEORY OF EQUATIONS. RULES FOR APPROXIMATION.

PLATE II.

24. In the last chapter the form of the parabolic curves, which are the loci of the equations of the form $y^m = x^n$, was examined only between the limits $x = 0$ and $x = 1$; in this chapter I shall consider the relative positions of the curves, and the way in which the curves bend, when x is greater than 1.

For this purpose draw lines parallel to Oy, intersecting Ox at A_1, A_2, A_3..., where $OA = 1$, $OA_2 = 2$, $OA_3 = 3$, &c. Fig. 3.

All the curves $y = x^2$, x^3,... pass through B where $AB = 1$.

On the line passing through A_2, measure distances

$$A_2P_2 = 4,\ A_2P_3 = 8,\ A_2P_4 = 16, \ldots$$

On the line A_3, measure $A_3Q_2 = 9$, $A_3Q_3 = 27$, ...

..................A_4,............$A_4Q_2 = 16$, $A_4Q_3 = 64$, ...

$OBP_2Q_2R_2$...is the curve $y = x^2$,

OBP_3Q_3 .... $y = x^3$,

OBP_4Q_4 .... $y = x^4$.

....

The tangents of the angles, which the curves make with the axis of x at B, are 2, 3, 4,...

PLATE II.

25. The forms of these curves shew that their curvatures rapidly diminish after passing the point (1, 1) although they remain convex to the axis of x, and that it is only by a strong effort of the imagination that it is possible to conceive what becomes of the curves, when x is made of any considerable magnitude.

We shall, in the investigation of the forms of curves, have to consider their general shapes at a great distance from the origin, and it will be readily seen from the results obtained here, that it would be impossible to represent on paper the proper proportions of the different parts of such curves; we are therefore obliged to content ourselves by indicating the direction of the bending of the curves at a great distance, leaving to the imagination what would become of the branches if extended in the proper proportion, on the same principle as in a raised map of a mountainous district, like that of Flintoft in Keswick, it is found to give a better idea of the form of the country to take a scale for the vertical heights differing very materially from the scale of horizontal distances.

26. The form of curves such as $y^2 = x^3$ in which the order of y is a fraction between 1 and 2, may be conceived by measuring, as ordinates, the approximate distances $A_2\pi = 2\frac{4}{5}$, $A_3\kappa = 5\frac{1}{5}$, $A_4\rho = 8$, $A_5\sigma = 11\frac{1}{5}$, $A_6\tau = 14\frac{5}{7}$, and $A_7\upsilon = 18\frac{1}{2}$, and drawing a curve touching the axis of x at the origin, passing through B at an angle whose tangent is $\frac{2}{3}$, and also through the points π, κ, ρ, σ, τ, υ.

It will be seen that such a curve opens out more rapidly than the former curves as x increases, and near the origin leaves the axis of x more rapidly than those curves do.

The figure which represents this curve in the more distant points is necessarily constructed with an unit too small to give an idea of the form near the origin.

Fig. 4. The shape of this curve, called the semicubical parabola, within the limits $x = 0$ and $x = 1$, is BOB', touching the lines BC, $B'C$ at B and B', where $OC = \frac{1}{3} OA$.

PLATE II.

27. To illustrate the use which it is intended to make of the forms of these curves near the origin and at a great distance from it, we will trace the curve $y = x^2 + x^3$.

Near the origin the approximate form of the curve is that of $y = x^2$; and, when x is very great, $y = x^3$, whose general form is that of the dotted curve; again when $x = -1$, $y = 0$, and if $x = -1 + \xi$, then $y = \xi(1-\xi)^2$; therefore, neglecting powers of ξ above the first near this point, $y = \xi = x + 1$ is a tangent. Fig. 5.

28. Although the Differential Calculus gives some advantages towards obtaining the particular points which appear in the form of the curve, yet it is not difficult to obtain them by the ordinary processes of Algebra.

For example, we observe in the figure a maximum ordinate near a, where the curve is parallel to the axis of x; if (α, β) be any point on the curve, let

$$x = \alpha + \xi, \quad y = \beta + \eta,$$

$$\beta + \eta = (\alpha + \xi)^2 + (\alpha + \xi)^3,$$

and $$\beta = \alpha^2 + \alpha^3;$$

therefore $$\eta = (2\alpha + 3\alpha^2)\,\xi + (1 + 3\alpha)\,\xi^2 + \xi^3 \,\ldots\ldots\ldots\ldots (1),$$

is the equation referred to axes through a, and $\eta = (2\alpha + 3\alpha^2)\,\xi$ is the tangent, which is parallel to the axis of x, if $\alpha = -\frac{2}{3}$, and $\beta = \frac{4}{27}$.

29. Again, there is a point of inflexion near b. If (α, β) be this point of inflexion b, since the curve must lie on opposite sides of the tangent at b, the term in (1) which involves ξ^2 must disappear;

therefore $$3\alpha + 1 = 0,$$

and the equation referred to b as origin, is

$$\eta = -\tfrac{1}{3}\xi + \xi^3,$$

where η is $>$ or $< -\frac{1}{3}\xi$, according as ξ is positive or negative.

PLATE II. The point of inflexion is $(-\frac{1}{3}, \frac{2}{27})$, and the inclination to Ox' is $\cot^{-1}3$.

EXAMPLES OF TRACING.

30. At this stage it will be useful to trace some curves in which the abscissa and ordinates are not involved with one another in any complicated manner, so that the methods already given are sufficient.

In such cases it will be seen, that there is rarely any necessity to enter minutely into the question of the direction of flexure from the tangent at any particular point, but the general run of the curve can be obtained by a small number of points and directions of tangents, combined with the consideration that a straight line cannot intersect the curve in more points than the greatest sum of the indices of x and y in any term.

Thus, it is impossible that there should be a point of inflexion or a cusp in a curve of the second degree; two cusps, or a multiple point of three branches, in a curve of the third degree.

Again, considering that a rectilinear asymptote to a curve is a straight line which joins two points at least at an infinite distance, it follows that a curve of the third degree may meet the asymptote at a finite distance in one point, and no more, one of the n^{th} degree in $n-2$ points and no more.

(1) $$y = x(x^2 - 1).$$

If $-x$ and $-y$ be written for x and y, the equation is not altered; therefore the curve is symmetrical in opposite quadrants, or the origin is a centre.

The principal points to examine are (0, 0) and (1, 0); near the origin, since $y = -x + x^3$, the curve is above the tangent when x is positive, and below when negative; near (1, 0) let $x = 1 + \xi$, therefore $y = 2\xi$; near (∞, ∞), $y = x^3$, represented by the dotted line in the figure. Fig. 6.

(2) $$cy^2 = x(x^2 + ax + b).$$

This curve is symmetrical with respect to Ox, it meets

PLATE II.

the axis of x at the origin, and where $x^2+ax+b=0$, and passes off to infinity in the form of $cy^2=x^3$; the shape near the origin is $cy^2=bx$.

i. When $x^2+ax+b\equiv(x-\alpha)(x-\beta)$, $\alpha>\beta$, Fig. 7.

$$\text{near } (\beta, 0),\ cy^2=\beta(\beta-\alpha)\,\xi,$$
$$\ldots\ldots\ (\alpha, 0),\ cy^2=\alpha\,(\alpha-\beta)\,\xi.$$

In the figure the dotted oval is the position of the oval when β is negative.

ii. When $x^2+ax+b\equiv(x-\alpha)^2$, Fig. 8.

$$\text{near } (\alpha, 0),\ cy^2=\alpha\xi^2.$$

iii. $x^2+ax+b\equiv(x-\alpha)^2+\beta^2$.

For one figure is drawn when $\alpha>\beta\sqrt{3}$, and one when $\alpha<\beta\sqrt{3}$. Figs. 9, 10.

When a and b vanish, all these curves degenerate into the asymptotic curve, viz. the semi-cubical parabola, $cy^2=x^3$.

It is useful to consider this degeneration, because it explains how it may happen that, when $y=0$, there are three values of x equal to 0, in the curve $cy^2=x^3$.

We may also see how the theory of equation assists us in drawing the curve in proper proportions.

Thus, for any value of y the three values of x have their algebraical sum the same, so that if nab be a tangent at a, Fig. 7. parallel to Ox, and the curve cuts the axis of x in α, β,

$$2na+nb=O\alpha+O\beta;$$

therefore if am, br be ordinates at a, b,

$$Om+\beta r=m\alpha,$$

and when α, β coincide, $bn=2am$. Fig. 8.

If nab, $n'a'b'$ be two tangents parallel to Ox, the distance between b and b' measured parallel to Ox is double the Fig. 9. distance between a and a'.

(3) $y=a\dfrac{(x-a)(x-3a)}{x.(x-2a)}$,

$x=0$, $\quad y=\infty$,

$x<a$, $\quad y$ is negative,

$x=a$, $\quad y=0$,

PLATE II.

$x > a < 2a,\quad y$ is positive,

$x = 2a,\quad y = \infty,$

$x > 2a < 3a,\quad y$ is negative,

$x = 3a,\quad y = 0,$

$x > 3a,\quad y$ is positive,

$x = \infty,\quad y = a,$

when x is negative, y is positive,

$$\text{near } (a, 0),\qquad y = a\,\frac{(x-a)(-2a)}{a(-a)} = 2(x-a),$$

$$\ldots\ldots\ (3a, 0),\qquad y = a\,\frac{2a(x-3a)}{3a^2} = \tfrac{2}{3}(x-3a),$$

Fig. 11. for the form, see the figure.

(4) $$y^2 = a^2\,\frac{(x-a)(x-3a)}{x(x-2a)}.$$

The curve is symmetrical with respect to Ox, and writing in the tabulation given above, y impossible for y negative, y real for y positive and $x = \infty$, $y = \pm a$,

$$\text{near } (a, 0),\quad y^2 = 2a(x-a),$$

$$\ldots\ldots\ (3a, 0),\ y^2 = \frac{2a}{3}(x-3a),$$

Fig. 12. we have the form given in the figure.

(5) $$(y^2 - 1)\,y = (x^2 - 4)\,x.$$

$-x$, $-y$ for x, y do not alter the equation, the curve is therefore symmetrical in opposite quadrants; hence, it is only necessary to examine the form for positive values of x,

$x = 0,\ y = 0,\ 1,\ -1,$

$x < 2,\ y$ negative > 1, numerically,

or positive < 1,

$x = 2,\ y = 0,\ 1,\ -1,$

$x > 2,\ y$ negative < 1,

or positive > 1,

$x = \infty,\ y = \infty,$

PLATE II.

near $(0, 0)$, $y = 4x$,

$(0, 1)$, $2(y-1) = -4x$,

$(0, -1)$, $2(y+1) = -4x$,

$(2, 0)$, $-y = 8(x-2)$,

$(2, 1)$, $2(y-1) = 8(x-2)$,

$(2, -1)$, $2(y+1) = 8(x-2)$,

near (∞, ∞), $y = x$, which meets the curve only at the origin, and at two points at an infinite distance. Fig. 13.

(6) $$y(y-1)(y-2) = x(x^2-1)(x-2).$$

The equation is made more symmetrical by transferring the origin to a point $(\frac{1}{2}, 1)$, becoming

$$y(y^2-1) = (x^2 - \tfrac{9}{4})(x^2 - \tfrac{1}{4}),$$

the curve is therefore symmetrical with respect to this axis Oy.

$x = 0$, $y(y^2-1) = \frac{9}{16}$.

$x < \frac{1}{2}$, y positive > 1, or negative < 1,

$x = \frac{1}{2}$, $y = 0$, or ± 1,

$x > \frac{1}{2} < \frac{3}{2}$, y positive < 1, or negative > 1,

$x = \frac{3}{2}$, $y = 0$, or ± 1,

$x > \frac{3}{2}$, y positive > 1, or negative < 1,

$x = \infty$, $y = \infty$,

y near 0, or ± 1, gives $y(y^2-1) = -y$, or $2(y \mp 1)$,

x near $\frac{1}{2}$, or $\frac{3}{2}$, gives $(x^2 - \frac{9}{4})(x^2 - \frac{1}{4}) = -2(x - \frac{1}{2})$, or $6(x - \frac{3}{2})$; from which the tangents can be found at $(\frac{1}{2}, 0)$, $(\frac{1}{2}, \pm 1)$, $(\frac{3}{2}, 0)$, $(\frac{3}{2}, \pm 1)$; also near (∞, ∞), $y^3 = x^4$.

The tangents are parallel to Oy, where $y = \pm \dfrac{1}{\sqrt{3}}$, and to Ox, where $x = \pm \frac{1}{2}\sqrt{5}$ and 0. Fig. 14.

This curve affords another illustration of the manner in which we may suppose a curve, whose equation is $y^3 = x^4$, to be drawn in order that, when $x = 0$, there may be three values of y, and, when $y = 0$, four values of x, since, by conceiving the unit of measurement to become very small

PLATE II.

compared with the size of the paper, the undulating part may be made as small as we please.

31. The following curve is given as an example by almost all the writers on the Differential Calculus, and deserves to be considered carefully as to the points which we have at present discussed.

$$y^4 - 96a^2y^2 + 100a^2x^2 - x^4 = 0,$$

it follows easily that

$$y^2 = 48a^2 \pm \sqrt{(x-8a)(x-6a)(x+6a)(x+8a)}.$$

Since the curve is symmetrical with respect to both axes, it need only be traced in the angle xOy,

$x = 0$, $y = 0$, or $a\sqrt{(96)} = a10\sqrt{(1 - \frac{1}{25})} = (10 - \frac{1}{5})\, a$ nearly,

$y = 0$, $x = 0$, or $10a$,

$x < 6a$, y is real,

$x = 6a$, $y = a\sqrt{(48)} = 7a\left(1 - \dfrac{1}{2 \times 49}\right) = a(7 - \frac{1}{14})$ nearly,

$x > 6a < 8a$, y is impossible,

$x = 8a$, $y = a(7 - \frac{1}{14})$ nearly,

$x = \infty$, $y = \infty$,

near the origin $y^2 = \frac{100}{96}x^2$;

$\therefore\ y = \pm(1 + \frac{1}{25})\, x$ nearly,

near $\{0, a\sqrt{(96)}\}$, let $y = a\sqrt{(96)} + \eta$;

$\therefore\ y^2 - 96a^2 = 2a\eta\sqrt{(96)}$;

$\therefore\ 2a\sqrt{(96)}.96a^2\eta + 100a^2x^2 = 0$,

retaining only the principal terms; therefore the curve is approximately a parabola, whose latus rectum is

$2a.10\,(1 - \frac{3}{50})$ nearly,

near $(10a, 0)$, let $x = 10a + \xi$;

$\therefore\ 100a^2 - x^2 = -20a\xi$;

$\therefore\ y^2 = -20a.\frac{100}{96}\xi$

$= -(20 + \frac{5}{6})\, a\xi$ nearly,

PLATE II.

near $\{6a, \sqrt{(48a)}\}$, let $y = \sqrt{(48)}\,a + \eta$, $x = 6a + \xi$;

$$\therefore\ 2\sqrt{(48)}\,a\eta = \pm\sqrt{(-2a\xi . 12a . 14a)};$$

$$\therefore\ \eta^2 = -\tfrac{7}{4}a\xi,$$

near $\{8a, \sqrt{(48)}\,a\}$, let $x = 8a + \xi$, $y = \sqrt{(48)}\,a + \eta$;

$$\therefore\ 2\sqrt{(48)}\,a\eta = \pm\sqrt{(2a\xi . 14a . 16a)};$$

$$\therefore\ \eta^2 = \tfrac{7}{3}a\xi,$$

$$\text{near } (\infty, \infty),\ y = \pm x,$$

and since these lines each meet the curve in only two points, viz. at the origin, at a finite distance, they meet the curve at two points infinitely distant, and they are therefore asymptotes. At A and D in the figure, the latus rectum of the approximate parabolas are a little less and a little greater than $2a$, and at B, C they are $\tfrac{7}{4}a$ and $\tfrac{7}{3}a$, and at the origin the approximate branches very nearly coincide with the asymptotes.

REPRESENTATION OF CHANGES OF FUNCTIONS.

32. One of the most useful applications of curves is to represent to the eye the changes which take place in any quantity in consequence of a change in a variable upon which the quantity depends; everybody is familiar with the curves representing the changes in the barometer and thermometer during the course of a day or month, in which the height of the barometer and degree of the thermometer, taken as ordinates, are functions of the time, taken as abscissa; also with isothermal lines; and, in terrestrial magnetism, with isoclinal and isodynamic lines. It is in fact much easier to follow the variations of the magnitude of ordinates to a curve, than to gather the same information from a table of numbers.

33. The changes of magnitude of the trigonometrical functions may be represented by taking each function for an ordinate of a curve, of which the angle is the abscissa,

LATE II.

The tangent is therefore parallel to the axes of x at every point for which $f'(\alpha)=0$, *i.e.* for every value of x, which is a root of the auxiliary equation $f'(x)=0$.

Now, it is obvious from the manner in which the curve must be drawn, that between every two distinct points in which the curve meets the axis of x there must be an odd number of bends, from or towards Ox, and therefore an odd number of points in which the curve must run parallel to the axis of x, where two bends must be supposed to coincide in the case of an ordinary point of inflexion, so that between every two roots of the equation $f(x)=0$ there are an odd number of roots of $f'(x)=0$.

Whence it may be deduced that, if $f(x)=0$ have two roots each equal to a, a will also be a root of $f'(x)=0$; which is also obvious, either from the fact, that the axis of x joins two points which are ultlmately coincident, or from the analysis, because

$$y=f(x)=(x-a)^2\,\phi(x);$$

$$\therefore \text{ if } x=a+\xi,\quad y=\xi^2.\phi(a),$$

is the first approximation, and this curve obviously touches the axis of x.

39. Other properties can be deduced immediately from the curve.

(1) If two values of x give values of $f(x)$ affected with opposite signs, an odd number of roots of $f(x)=0$ must lie between these values.

(2) Every equation of an even degree must have an even number of roots, since the curve terminates in both directions on the same side of the axis Ox.

(3) Every equation of an odd degree must have at least one real root of the sign contrary to that of the last term.

For the curve cuts the axis of y on the positive or negative side, as the last term is positive or negative; if

E

21.

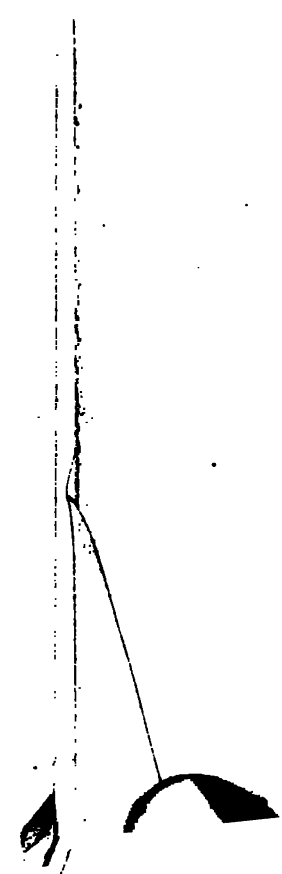


PLATE II.

t must cut the axis of x in the negative direction, curve goes off to infinity in the angle $x'Oy'$, and

ery function of x, which equated to zero has no must be invariable in sign.

o find a point of inflexion in the curve $y=f(x)$.
, β) be a point of inflexion, transfer the origin to , by writing $\alpha+\xi$, $\beta+\eta$ for x, y.
ding as in Art. 3, (4),

$$=f(\alpha)+f'(\alpha)\xi+\tfrac{1}{2}f''(\alpha)\xi^2+\tfrac{1}{6}f'''(\alpha)\xi^3+\ldots,$$

$=f(\alpha)$, (α, β) being a point on the curve,

$\eta=f'(\alpha)\xi$ is the tangent, and, in the curve, the $-f'(\alpha)\xi$, being the difference of the ordinates for and tangent, must change sign when ξ changes refore $f''(\alpha)$ must vanish when α is the abscissa of inflexion.

nce in Art. 3, theorem (4), it is shewn that $f''(x)$ from $f'(x)$ in the same manner as $f'(x)$ from $f(x)$, ve $y=f'(x)$ be traced, and the values of x found, ling to which the curve runs parallel to Ox, the ling ordinates, produced if necessary, will pass he points of inflexion of $y=f(x)$.
is consideration, and that the same curve cuts the for all values of x for which the original curve l to Ox, it is easy to give the general form of $y=f'(x)$; part of it is traced by a fainter line ure, in which it may be observed, that when $f(x)$ ng with x, $f'(x)$ is positive, and *vice versâ*. Fig. 21.

$f''(\alpha)$ be not $=0$, the curve has a parabolic form.
$=0$ is a necessary, though not a sufficient condition it of inflexion, the sufficient condition is, that the in $\eta-f'(\alpha)\xi$ shall be of an odd degree.

PLATE III.

43. The necessary condition for a point of inflexion may also be obtained thus.

If ϕ be the inclination of the tangent at a point where $x=\alpha$, $\tan\phi = f'(\alpha)$, and $f'(\alpha+\xi) = f'(\alpha) + f''(\alpha)\,\xi$, as near as we please by diminishing ξ; therefore since, in passing through a point of inflexion, ϕ increases and afterwards diminishes, or *vice versâ*, $f'(\alpha+\xi) - f'(\alpha)$ has the same sign, whether ξ be positive or negative, which is impossible unless $f''(\alpha)=0$.

44. As an example, take the curve

$$\begin{aligned} y &= x(x-1)(x^2-1) \\ &= x^4 - x^3 - x^2 + x. \end{aligned}$$

The tangents are parallel to the axis of x, where

$$4x^3 - 3x^2 - 2x + 1 = 0,$$

which gives $\quad x = 1$ and $x = \frac{3}{8}$, or $-\frac{5}{8}$ nearly,

and there may be points of inflexion, where

$$12x^2 - 6x - 2 = 0,$$

Fig. 1. which gives $\quad x = \dfrac{\pm\sqrt{(33)}+3}{12} = \frac{35}{48}$, or $-\frac{11}{48}$ nearly.

The fainter line is the curve $y=f'(x)$, whose maximum and minimum ordinates pass through the points of inflexion of the given curve.

45. The nature of the roots of an equation can be frequently discovered by the use of the intersection of simple curves; an example or two will be sufficient to shew the method.

Consider the cubic $x^3 - qx + r = 0$, the roots of the equation are the abscissæ of the points of intersection of

Fig. 2. the parabola

$$y = x^2 \text{ ....(1)},$$

and the hyperbola $\quad xy - qx + r = 0 \text{ ....(2)},$

or $\quad x(q-y) = r.$

PLATE III. Fig. 2.

The figure shews how the hyperbola, which remains of constant magnitude if r be constant, changes its position, denoted by 1, 2, 3, as q increases from some negative value, moving along the axis of y as an asymptote, and first touching the parabola with its lower branch, then intersecting it in two points, additional to the one in which the higher branch necessarily cuts the parabola.

46. The condition of equal roots corresponding to the curves having a common tangent would be deduced as follows:

The equation of the tangent to the hyperbola at (α, β) is $(\beta - q)(x - \alpha) + \alpha(y - \beta) = 0$, and since it touches the parabola also, we know that when $y = 0$ $x = -\alpha$,

$$\therefore\ -2\alpha(\beta - q) - \alpha\beta = 0,$$

$$\therefore\ \beta = +\tfrac{2}{3}q = \alpha^2 \text{ by (1)},$$

and $$\alpha(q - \tfrac{1}{3}q) + r = 0 \text{ by (2)},$$

$$\therefore\ r^2 = \tfrac{4}{27}q^3.$$

47. Another way would be to consider the roots as the abscissæ of the points of intersection of the curve, $y = x^3$, with the straight line $y - qx + r = 0$.

The straight line always meets the curve in one real point, and the straight line turns round as q increases, until it touches, in which case two roots are equal, and afterwards intersects in two more real points.

The equation of the tangent at (α, β) is

$$y - \beta = 3\alpha^2(x - \alpha),$$

or $$y = 3\alpha^2 x - 2\alpha^3,$$

therefore if $y - qx + r = 0$ be a tangent, we have

$$3\alpha^2 = q \text{ and } 2\alpha^3 = r,$$

and $$\frac{r^2}{4} = \frac{q^3}{27}.$$

PLATE III.

48. The roots of the biquadratic

$$x^4 + qx^2 + rx + s = 0,$$

are given by the points of intersection of the two parabolas

$$y = x^2,$$

and $$y^2 + qy + rx + s = 0,$$

or $$(y + \tfrac{1}{2}q)^2 = -r\left(x + \frac{q^2 - 4s}{4r}\right).$$

Fig. 3. The figure shews how the cases of real or impossible or equal roots may arise, from the relative positions of the parabolas, 1, 2, 3 and 4.

49. In the course of this work, many examples will occur of the practical advantage of this method in determining the number of real roots of an equation, as well as in roughly fixing their values.

RULES FOR APPROXIMATION.

50. It will be found useful to have some simple method of approximating to the value of a quantity to any order of standard small quantities which may be required. Such a method can be established as follows:

If $$y = a + \alpha f(y) + \alpha^2\phi(y) + \ldots,$$

where α is the standard small quantity which does not appear in the functions $f(y)$, $\phi(y)$, representing the coefficients of the different powers of α.

Suppose that ordinary algebraical processes give

$$f(y+k) = f(y) + f_1(y)\,k + f_2(y)\,k^2,$$

substituting the general value of y in the functions,

$$\begin{aligned} y &= a + \alpha f[a + \{\alpha f(y) + \alpha^2\phi(y) + \ldots\}] \\ &\quad + \alpha^2\phi[a + \{\alpha f(y) + \ldots\}] + \ldots \\ &= a + \alpha[f(a) + f_1(a)\{\alpha f(y) + \alpha^2\phi(y) + \ldots\} \\ &\qquad + f_2(a)\{\alpha f(y) + \ldots\}^2 + \ldots] \\ &\quad + \alpha^2[\phi(a) + \phi_1(a)\{\alpha f(y) + \ldots\} + \ldots]. \end{aligned}$$

In the first approximation we neglect α, whence $y = a$.

In the second, we neglect powers of α above the first;

$$\therefore\ y = a + \alpha f(a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

In the third, we neglect powers of α above the second,

$$\therefore\ y = a + \alpha\{f(a) + f_1(a)\,\alpha f(a)\} + \alpha^2\phi(a);$$

$$\therefore\ y = a + \alpha f\{a + \alpha f(a)\} + \alpha^2\phi(a) \ldots\ldots\ldots\ldots\ldots (2)$$

will produce a correct third approximation.

The following rule may be obtained from (1) and (2):

To obtain the first approximation, neglect α; to obtain the second approximation, substitute in the coefficient of α the result of the first approximation and neglect terms in α^2; to obtain the third approximation, substitute in the coefficient of α^2 the result of the first, and in the coefficient of α that of the second approximation, and neglect terms in α^3, and so on.

51. We have already employed this method in some simple cases, but it is well to have it stated in its more general form. In the following examples I have proceeded to many terms to show that there is no great difficulty in the application.

(1) To expand $\tan x$ in ascending powers of x.

By Gregory's series

$$x = \tan x - \tfrac{1}{3}\tan^3 x + \tfrac{1}{5}\tan^5 x - \tfrac{1}{7}\tan^7 x + \ldots;$$

$$\therefore\ \tan x = x + \tfrac{1}{3}\tan^3 x - \tfrac{1}{5}\tan^5 x + \tfrac{1}{7}\tan^7 x - \ldots.$$

First approximation $\quad \tan x = x$.

Second approximation $\quad \tan x = x + \frac{1}{3}x^3$.

For third approximation

$$\tan x = x + \tfrac{1}{3}(x + \tfrac{1}{3}x^3)^3 - \tfrac{1}{5}x^5 = x + \tfrac{1}{3}x^3 + (\tfrac{1}{3} - \tfrac{1}{5})\,x^5.$$

For the fourth approximation

$$\tan x = x + \tfrac{1}{3}(x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5)^3$$
$$- \tfrac{1}{5}(x + \tfrac{1}{3}x^3)^5 + \tfrac{1}{7}x^7$$

PLATE III.

$$= x + \tfrac{1}{3}x^3\left\{1 + x^2 + \tfrac{2}{5}x^4 + \tfrac{1}{3}\left(x^2 + \tfrac{2}{5}x^4\right)^2 + \ldots\right\}$$
$$- \tfrac{1}{5}x^5\left(1 + \tfrac{5}{3}x^2\right) + \tfrac{1}{7}x^7$$
$$= x + \tfrac{1}{3}x^3 + \left(\tfrac{1}{3} - \tfrac{1}{5}\right)x^5 + \left\{\tfrac{1}{3}\left(\tfrac{2}{5} + \tfrac{1}{3}\right) - \tfrac{1}{3} + \tfrac{1}{7}\right\}x^7$$
$$= x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7 + \ldots\ldots$$

(2) If $$r = a(1 + e\cos u),$$
$$u = m + e\sin u.$$

To expand r to e^3.

First approximation $\quad u = m.$

Second approximation $\quad u = m + e\sin m;$

$$\therefore\ \sin u = \sin m + \cos m \,.\, e\sin m.$$

Third approximation $\quad u = m + e\sin m + e^2\sin m\cos m$
$$= m + \alpha;$$
$$\cos u = \cos m\left(1 - \tfrac{1}{2}\alpha^2\right) - \sin m \,.\, \alpha$$
$$= \cos m - \sin m\left(\sin m + e^2\sin m\cos m\right)$$
$$- \tfrac{1}{2}\cos m \,.\, e^2\sin^2 m$$
$$= \cos m - e\sin^2 m - \tfrac{3}{2}e^2\sin^2 m\cos m;$$
$$\therefore\ r = a\left(1 - e\cos m + e^2\sin^2 m + \tfrac{3}{2}e^3\sin^2 m\cos m\right).$$

(3) To find y in ascending powers of x, when
$$6x^7 - 2x^5y^2 - a^3x^2y^2 + 4a^3x^3y + 2a^5x^2 - 3a^5xy + a^5y^2 = 0,$$
y being of the same order as x.

The first approximation gives
$$y^2 - 3xy + 2x^2 = 0;$$
$$\therefore\ y = x, \text{ or } 2x.$$

i. Commencing with $y = x$ for the second approximation,
$$y = x - \frac{4a^3x^3y - a^3x^2y^2}{a^5(y - 2x)}$$
$$= x + \frac{3x^3}{a^2}.$$

For the third, $y = x - \frac{1}{a^2} \cdot \frac{4x^3y - x^2y^2}{y - 2x} - \frac{6x^7 - 2x^5y^2}{a^5(y-2x)}$

$$= x + \frac{x^3}{a^2} \cdot \frac{4\left(1 + \frac{3x^2}{a^2}\right) - \left(1 + \frac{3x^2}{a^2}\right)^2}{1 - \frac{3x^2}{a^2}} + \frac{4x^6}{a^5}$$

$$= x + \frac{x^3}{a^2}\left(3 + \frac{6x^2}{a^2}\right)\left(1 + \frac{3x^2}{a^2}\right) + \frac{4x^6}{a^6}$$

$$= x + \frac{3x^3}{a^2} + \frac{15x^5}{a^4} + \frac{4x^6}{a^5} + \dots .$$

ii. Commencing with $y = 2x$ for the second approximation,

$$y = 2x - \frac{1}{a^2}\,\frac{x^2y\,(4x - y)}{y - x}$$

$$= 2x - \frac{4x^3}{a^2}.$$

For the third,

$$y = 2x - \frac{1}{a^2}\,\frac{x^2\left(4x^2 - \overline{y - 2x}\,|^2\right)}{y - x} - \frac{2x^5(3x^2 - y^2)}{a^5(5 - x)}$$

$$= 2x - \frac{1}{a^2}\,\frac{x^2 . 4x^2}{x\left(1 - \frac{4x^2}{a^2}\right)} + \frac{2x^6}{a^5}$$

$$= 2x - \frac{4x^3}{a^2} - \frac{16x^5}{a^4} + \frac{2x^6}{a^5}.$$

(4) To approximate to the solutions of the equation $\tan x = x$, the unit being the unit of circular measure.

Draw the curve $y = \tan x$, and let the straight line, $y = x$, meet the curve in the points $\dots P'$, O, P, Q, ... touching it at O. (Fig. 17, Plate II).

The figure shews that, since the branches of the curve are all similar, the abscissæ of the points P, Q, R, &c. are nearer and nearer to the values $\frac{3}{2}\pi$, $\frac{5}{2}\pi$, $\frac{7}{2}\pi$, &c.

PLATE III.

If, therefore, $x = \frac{2n+1}{2}\pi - z$, where n is any positive integer, z is a small quantity, and substituting in the equation,

$$\tan(n\pi + \tfrac{1}{2}\pi - z) = (2n+1)\,\tfrac{1}{2}\pi - z.$$

For $(2n+1)\,\frac{1}{2}\pi$ write $\frac{1}{\alpha}$, α is small, since its greatest value, when $n = 1$, is $\frac{2}{3\pi} = \frac{7}{33}$ nearly;

$$\therefore\ \cot z = \frac{1}{\alpha} - z,$$

and
$$\tan z = \frac{\alpha}{1 - \alpha z}.$$

By Gregory's series

$$z = \alpha\,(1 - \alpha z)^{-1} - \tfrac{1}{3}\alpha^3\,(1 - \alpha z)^{-3} + \tfrac{1}{5}\alpha^5\,(1 - \alpha z)^{-5} - \dots$$

The first approximation gives $z = \alpha$; the second gives

$$z = \alpha\,(1 + \alpha z) - \tfrac{1}{3}\alpha^3, \text{ or } z = \alpha + \tfrac{2}{3}\alpha^3.$$

The third approximation gives

$$\begin{aligned} z &= \alpha\,(1 + \alpha z + \alpha^2 z^2) - \tfrac{1}{3}\alpha^3\,(1 + 3\alpha z) + \tfrac{1}{5}\alpha^5 \\ &= \alpha + \alpha^2\,(\alpha + \tfrac{2}{3}\alpha^2) + \alpha^5 \\ &\quad - \tfrac{1}{3}\alpha^3 - \alpha^5 + \tfrac{1}{5}\alpha^5 \\ &= \alpha + \tfrac{2}{3}\alpha^3 + \tfrac{13}{15}\alpha^5. \end{aligned}$$

The solutions are 0, and

$$\pm\left(\frac{1}{\alpha} - \alpha - \tfrac{2}{3}\alpha^3 - \tfrac{13}{15}\alpha^5 - \dots\right),$$

in which $\alpha = \frac{2}{3\pi}, \frac{2}{5\pi}, \frac{2}{7\pi}, \dots$ successively.

The approximate solutions for the first value give 257° 27′ 12″. The only error introduced by omitting the fourth term amounts to little more than 8″.

III.

(1) Find the point at which the curve $y = x(x^2 - y)$ runs parallel to the axis of x.

(2) Shew that the distance between the two tangents to the curve $y = x(x^2 - 1)$, drawn parallel to the axis of x, is $\frac{4}{9}\sqrt{3}$.

(3) Find the shape of the curve $y^2 - 2y = x^4 - x^2$ at the points of intersection with Ox, and at an infinite distance.

(4) Shew that, if the form of the oval part of the curve $y^2 = x(x-1)(x-2)$ be represented near the two vertices by two parabolas, having their axes coincident with Ox, the latus rectum of one is double that of the other.

Trace the curves:

(5) $y^2 = x^3 + x^4$.

(6) $y^2 = x^3 - x^5$.

(7) $y^5 = (x^2 - 1)^2$.

(8) $y^6 = (x-1)^3 x$.

(9) $y^3 = (x-1)x^4$.

(10) $y^2 - y = x^3 - x^2$.

(11) Shew that there are points of inflexion at $(2, -16)$, $(4, 0)$ in the curve $y = x^4 - 12x^3 + 48x^2 - 64x$.

(12) Find the point of inflexion in the curve

$$y^2 = x\{(x-1)^2 + \tfrac{1}{4}\},$$

and trace the curve.

(13) Trace the curve whose equation is

$$xy^2 = (x-a)(x-b)(x-c),$$

a, b, c being real quantities.

(14) Shew by a curve the changes of sign and magnitude of $\sin x - \sin 2x$.

(15) Trace the curve $y = \sin x + \sin 2x - \sin 3x$.

PLATE III.

(16) Trace the curve $y = mx + c \sin \frac{\pi x}{a}$, when ma is greater than, equal to, or less than πc, and shew that the sum of the cotangents of the angles, at which the curve cuts any two ordinates whose distance is a, is constant.

(17) Trace the curve $xy^2 - 2yx + y - 1 = 0$, and employ it to shew that there is only one real root of the equation

$$x^7 - 2x^4 + x^3 - 1 = 0.$$

(18) Shew, by the intersection of two curves, that there are two imaginary and three real roots of the equation

$$x^5 - 4x^2 + 3 = 0.$$

(19) Prove that there cannot be more than three real roots of the equation

$$x^7 - a^5x^2 + c^7 = 0,$$

if c be positive, nor more than one if c be negative.

(20) Find, by means of two curves, properties of the roots of the equations

$$x^4 - 15a^3x + 14a^4 = 0, \text{ and } x^5 - 2a^2x^3 + 3a^5 = 0.$$

(21) Expand y in ascending powers of x, for three terms, when x is small, in the case of the two curves

$$x^3 + y^3 - 3axy = 0, \text{ and } x^4 - 3xy^2 + 2y^3 = 0.$$

(22) Expand y in descending powers of x, when x is very large, in the case of the two curves

$$x^4 - y^4 - 4ax^2y - 2ay^2 = 0, \text{ and } yx^2 - 2ax^2 - a^2y = 0.$$

(23) Solve the equation $\cos\theta = \theta$ approximately, having given that $\cos\alpha = \alpha + \beta$, when β is a small quantity.

(24) Approximate to the solutions of the equation $4x \tan x = \pi$, in which the unit is the unit of circular measure.

CHAPTER IV.

FORMS OF CURVES IN THE NEIGHBOURHOOD OF THE ORIGIN. SIMPLE TANGENTS. DIRECTION AND AMOUNT OF CURVATURE. MULTIPLE POINTS OF TWO BRANCHES. CURVATURE OF BRANCHES AT MULTIPLE POINTS. MULTIPLE POINTS OF HIGHER ORDERS.

PLATE III.

55. IN this and the following chapter I intend to discuss the forms of curves at particular points at a finite distance, when their equations are of a more complicated form.

For this purpose, it will be sufficient, as has already been mentioned, to consider the forms in the neighbourhood of the origin, since by transformation of coordinates, any point may be made the origin.

In order to trace a curve, we must know, at all points which have any peculiarity, the tangent, the side of the tangent on which the curve lies, and, in some cases, the rapidity with which the curve deflects itself from the tangent, *i.e.* the degree of curvature.

Although, in a great many cases the direction in which the curve bends from the tangent at any particular point, may appear from consideration of other known portions of the curve, we must be in possession of methods of determining this when required, however troublesome the operation may be.

In illustrating these methods, I have given the forms of many of the curves throughout, although it has at present been shewn only very generally how the infinite branches can be determined. The student should, at all events, obtain the directions of the curves at particular points, deferring the complete tracing, until the chapter on asymptotes has been read. I have, however, avoided, as much as possible, any complicated forms of infinite branches.

PLATE III.

TRACING CURVES.

56. Consider then, when a curve passes through the origin, and a point $P(x, y)$ is taken near the origin, on the branch which passes through it, that, as P is supposed to move towards O, x and y at some stage must diminish together, and that they ultimately vanish simultaneously.

One of the three following cases must therefore occur, as P describes the last small arc of the curve.

(1) x and y may be of the same order of small quantities.

(2) x may be small compared with y, in other words, $x : y$ may be a small ratio which ultimately vanishes.

(3) y may be small compared with x.

I think it best to consider these cases separately, and I shall in this chapter examine the cases in which x and y are of the same order of small magnitudes, so that an expression like $ax^2 + bxy + cy^2$ has every term of the second order, and is itself of the second order; we may observe that $ax^2 + bxy + cy^3 + dx^2y^2$ is also of the second order, if a, b, c, d be finite quantities.

57. Let the equation of a curve, supposed rationalized, be arranged in a series of homogeneous functions of x and y, in the form

$$u_1 + u_2 + u_3 \dots = 0,$$

where u_s denotes a homogeneous function of s demensions in x and y; so that there being no term independent of x and y, the curve is here supposed to pass through the origin.

SIMPLE TANGENT.

58. Consider first the case in which u_1, the function of the first degree exists, and let $u_1 \equiv ax + by$, and at present consider a and b to be finite quantities, $u_1 = 0$ is the first approximation to the relation which must exist between x and y for the part of the curve near the origin, for the whole equation may plainly be written

$$a(1 + \alpha)x + b(1 + \beta)y = 0,$$

where, by diminishing x and y, α and β may be made as small as we please.

Thus, if the equation were

$$ax + by + cx^2 + dxy + ey^2 = 0,$$

it might be written

$$a\left(1 + \frac{c}{a}x + \frac{d}{a}y\right)x + b\left(1 + \frac{e}{a}y\right)y = 0.$$

Hence, $u_1 = 0$ represents the direction of the curve near the origin.

59. It can be shewn as follows that no straight line can be drawn, which, near the origin, lies so close to the curve as $u_1 = 0$.

For if (x, y) be any point in the curve, the perpendicular from this point on $u_1 = 0$ is $\dfrac{ax + by}{\sqrt{(a^2 + b^2)}} = -\dfrac{u_2 + u_3 + \ldots}{\sqrt{(a^2 + b^2)}}$, which is of the second order of small quantities, whereas the perpendicular on any other line $lx + my$ is $\dfrac{lx + my}{\sqrt{(l^2 + m^2)}}$, which is of the first order.

DIRECTION AND AMOUNT OF CURVATURE.

60. The next step is to discover in what direction the curve bends after leaving the point of contact; for this purpose we proceed to the next approximation, by taking into account the terms of the next order; the form of the curve is *generally* given more nearly by $u_1 + u_2 = 0$, if there be any terms at all of the second degree.

This is the equation of a conic. At present we shall not consider the case in which this conic is two straight lines, in which case it will be seen that $u_1 + u_2 = 0$ is not the next approximation.

Such a conic may be called a conic of curvature, since it has the same curvature as the given curve in the neighbourhood of the origin.

PLATE III.

61. The proof that the two curves have the same curvature may be given thus.

Fig. 4. Let OT be the tangent $u_1 = 0$,

OP' the conic $u_1 + u_2 = 0$,

OP the curve $u_1 + u_2 + u_3 + \ldots = 0$.

Draw TPP' perpendicular to OT, and let (x, y), (x', y') be the points P, P'; it may be shewn, as in Art. 59, that

$$PT : P'T = u_2 + u_3 + \ldots : u'_2,$$

and since $x : y = b : -a = x' : y'$ ultimately, $u_2 : u'_2$ is a ratio of equality ultimately; therefore PT and $P'T$ are ultimately equal, which is the test of the curvature being the same in the two curves.

62. That there are an infinite number of conics which have the property of coinciding with the curve to within quantities of the third order, may be seen as follows.

For, $u_1 = ax + by = -u_2 - u_3 \ldots$ is a quantity of the second order, for points on the curve near the origin; therefore $(ax + by)(\lambda x + \mu y)$ is of the third order, λ, μ being any constants arbitrarily chosen,

$$\therefore\ u_1 + u_2 + (\lambda x + \mu y)\, u_1 = 0$$

is a conic which differs from the curve by a quantity of the third order.

63. The particular conic which is the circle of curvature can thus be found by giving proper values to λ, μ; if

$$u_2 \equiv cx^2 + dxy + ey^2,$$

making the coefficient of $xy = 0$, and those of x^2 and y^2 equal

$$\lambda b + \mu a + d = 0,$$

and $$\lambda a + c = \mu b + e = \rho \text{ suppose};$$

$$\therefore\ b\,\frac{\rho - c}{a} + a\,\frac{\rho - e}{b} + d = 0,$$

$$\rho = \frac{b^2c - abd + a^2e}{a^2 + b^2}.$$

PLATE III.

The equation of the circle is $x^2+y^2+\dfrac{ax+by}{\rho}=0$, the radius $=\frac{1}{2}\dfrac{\sqrt{(a^2+b^2)}}{\rho}=\frac{1}{2}\dfrac{(a^2+b^2)^{\frac{3}{2}}}{b^2c-abd+a^2e}$.

64. The diameter of curvature may also be found directly, without previously finding the equation of the circle.

For by Art. 23, the diameter of curvature is the limit of

$$\begin{aligned}\frac{OP^2}{PT} &= \text{limit} \pm \frac{x^2+y^2}{u_1}\sqrt{(a^2+b^2)}\\ &= \text{limit} \mp \frac{x^2+y^2}{u_2}\sqrt{(a^2+b^2)}\\ &= \pm\frac{(a^2+b^2)^{\frac{3}{2}}}{b^2c-abd+a^2e}, \text{ Art. 3, (3), since } \frac{x}{b}=\frac{y}{-a} \text{ ultimately.}\end{aligned}$$

65. If u_2 does not appear in the equation, u_1 instead of being of the second is of the third order, thus there is no conic of curvature, since the deflection from the tangent for the curve is infinitely less than that of any conic which can be drawn, the deflection in a conic being of the second order.

Compare the curves $O\alpha_2B$, and $O\alpha_3B$ in fig. 13, Plate I.

The approximate curve $u_1+u_3=0$ has a point of inflexion at O, since the perpendicular from (x, y) on $u_1=0$, which varies as u_3, changes sign in passing through the origin.

66. The following examples will shew how to apply these properties:

(1) To find the tangent to the curve $y^r=x^s$ at the point (1, 1).

For x and y write $1+\xi$ and $1+\eta$, and arrange in homogeneous functions; the first term is $r\eta-s\xi$, which gives the equation of the tangent

$$r(y-1)=s(x-1).$$

PLATE III.

(2) To find the tangents to the curve $y = \frac{(x-1)(x-3)}{x-2}$ at the points (1, 0), (3, 0). For x write $1+\xi$; therefore the first approximation gives $y = \xi \frac{1-3}{1-2} = 2\xi$,

or $\qquad y = 2(x-1)$ is the equation of the first tangent,

similarly $y = 2(x-3)$ .... second tangent.

(3) To find the tangent and circle of curvature at a point (α, β) of the curve $ax^2 + by^2 = 1$.

For x, y write $\alpha+\xi$, $\beta+\eta$,

$$a(\alpha+\xi)^2 + b(\beta+\eta)^2 = 1,$$

$$a\alpha^2 \quad + \quad b\beta^2 \quad = 1;$$

$$\therefore\ 2(a\alpha\xi + b\beta\eta) + a\xi^2 + b\eta^2 = 0,$$

whence $a\alpha\xi + b\beta\eta = 0$ is the equation of the tangent referred to the new axes.

Since, near the origin, $a\alpha\xi + b\beta\eta$ is of the second order,

$$2(a\alpha\xi + b\beta\eta) + a\xi^2 + b\eta^2 - (a\alpha\xi + b\beta\eta)(\lambda\xi + \mu\eta) = 0$$

is a curve which differs from the given curve by a quantity of the third order, and this is a circle if λ, μ satisfy

$$a(1-\alpha\lambda) = b(1-\beta\mu) = \rho, \text{ and } a\alpha\mu + b\beta\lambda = 0,$$

whence $$a\alpha\frac{b-\rho}{b\beta} + b\beta\frac{a-\rho}{a\alpha} = 0,$$

$$\therefore\ \rho = \frac{ab(a\alpha^2 + b\beta^2)}{a^2\alpha^2 + b^2\beta^2} = \frac{ab}{a^2\alpha^2 + b^2\beta^2},$$

and the equation of the circle of curvature becomes

$$\xi^2 + \eta^2 + \left(\frac{a\alpha^2}{b} + \frac{b\beta^2}{a}\right)(2a\alpha\xi + 2b\beta\eta) = 0,$$

the square of whose radius is $\frac{(a^2\alpha^2 + b^2\beta^2)^3}{a^2b^2}$.

(4) To find the locus of the centre of the circle of curvature in the last curve.

Let $x-\alpha$, $y-\beta$ be written for ξ, η in the last equation.

The coefficient of x is

$$-2a\left\{1-\frac{a^2\alpha^2}{b}-b\beta^2\right\}=-2a\left(a\alpha^2-\frac{a^2}{b}\alpha^2\right);$$

therefore the coordinates of the centre are

$$\frac{a}{b}(b-a)\,\alpha^3 \text{ and } \frac{b}{a}(a-b)\,\beta^3 \text{ and } a\alpha^2+b\beta^2=1;$$

therefore, writing

$$\alpha=\left\{\frac{xb}{a(b-a)}\right\}^{\frac{1}{3}} \text{ and } \beta=\left\{\frac{ya}{b(a-b)}\right\}^{\frac{1}{3}},$$

$$\left(\frac{x}{\sqrt{a}}\right)^{\frac{2}{3}}+\left(\frac{y}{\sqrt{b}}\right)^{\frac{2}{3}}=\left(\frac{b-a}{ab}\right)^{\frac{2}{3}},$$

which is the equation of the locus.

(5) To find the diameter of the circle of curvature at the point (a, b) of the curve whose equation is

$$y(b^2-y^2)=x^2(a-x).$$

Let $x=a+\xi,\ y=b+\eta,\ -(b+\eta)(2b\eta+\eta^2)=-(a+\xi)^2\xi,$

or $$2b^2\eta-a^2\xi+3b\eta^2-2a\xi^2+\ldots=0.$$

The diameter of curvature $=$ limit $\dfrac{(\xi^2+\eta^2)\sqrt{(a^4+4b^4)}}{2b^2\eta\sim a^2\xi}$,

and $$\frac{\xi}{2b^2}=\frac{\eta}{a^2} \text{ ultimately};$$

therefore the diameter is $\dfrac{(a^4+4b^4)^{\frac{3}{2}}}{3ba^4\sim 8ab^4}=\dfrac{(a^4+4b^4)^{\frac{3}{2}}}{ab\,(3a^3-8b^3)}$.

COR. If $3a^3=8b^3$ the diameter is infinite.

In this case $2b^2\eta-a^2\xi=0$ is true to quantities of the second order; therefore the third approximation gives

$$2b^2\eta-a^2\xi+\eta^3-\xi^3=0,$$

or $$2b^2\eta=a^2\xi+\xi^3\left(1-\frac{a^6}{8b^6}\right)=a^2\xi+\frac{1}{9}\xi^3,$$

where $\dfrac{a^2}{2b^2}=\dfrac{25}{26}$ nearly,

PLATE III.

shewing that (a, b) is a point of inflexion. The form will be found in a succeeding chapter.

67. The preceding is the general case of a simple tangent; we can proceed by a similar method of approximation, when singularities occur in the forms of the functions succeeding u_1.

It will be sufficient to take, as an instance, the case mentioned above, Art. 60, in which the conic $u_1 + u_2 = 0$, becomes two straight lines, which happens when $u_2 \equiv u_1 v_1$.

68. In this case, the equation of the curve being

$$u_1 + u_1 v_1 + u_3 + \ldots = 0,$$

the tangent of the curve is $u_1 = 0$, and the next approximation to the form of the curve near the origin is given by

$$u_1 + \frac{u_3}{1 + v_1} = 0,$$

or

$$u_1 + u_3 = 0,$$

shewing that the distance of a point in the curve from the tangent, which varies as u_1, is ultimately of the third order, and, consequently, that $u_1 v_1$ is of the fourth order; thus, for points in the branch through the origin, u_3 ranks before u_2, shewing, as was stated in Art. 60, that $u_1 + u_2 = 0$ was not the next approximation.

Since u_3 changes sign, as we pass through the origin, the curve has a point of inflexion at the origin.

69. Take, as an example of such a form of u_2, the curve

$$a^3 (y + x) - 2a^2 x (y + x) + x^4 = 0.$$

The approximate form of the branch through the origin is given by

$$a^3 (y + x) + x^4 = 0,$$

which shows that the curve bends towards the negative end of Oy, and that its distance from the tangent is of the fourth order.

PLATE III.

It may be shewn that it cuts the axis of x at four points, by the discussion of the equation in the form

$$y=\frac{x(x-a)(x^2+ax-a^2)}{a^2(2x-a)},$$

and that the angles of inclination are $\pm 45^\circ$ and $\pm \tan^{-1}\sqrt{5}$.

The asymptotes are $2x=a$ and $2a^2y=x^3$. These considerations are sufficient for the general shape given in the figure. Fig. 4.

MULTIPLE POINT OF TWO BRANCHES.

70. I proceed next to the case of two branches through the origin. Such points occur when the rationalized equation commences with a function of the second degree, thus

$$u_2+u_3+u_4+\ldots=0,$$

in which u_2 may be of the form I. v_1w_1, II. v_1^2, III. $v_1^2+w_1^2$.

I. $u_2\equiv v_1w_1$.

71. The tangents to the curve are given by $v_1=0$, $w_1=0$; let $v_1\equiv ax+by$, and suppose a and b different from zero, so that x and y are of the same order of magnitude near the origin, the second approximation is obtained from $v_1+\frac{u_3}{w_1}=0$, which may be written $v_1+\frac{u_3}{w_1x^2}.x^2=0$, and, since $\frac{u_3}{w_1x^2}$ is a fraction whose numerator and denominator are each of the third degree of small quantities, we may write b for x and and $-a$ for y, Art. 3, (3), for the first approximation to the value of the coefficient of x^2; in fact, if $u_3\equiv f(x, y)$, $\frac{u_3}{w_1x^2}$ differs from $\frac{f(b, -a)}{(a'b-b'a)b^2}=A$ by a quantity of the first order, and the branch of the curve is represented nearly by $v_1+Ax^2=0$, and lies on the same side of the tangent on each side of the origin.

The process is practically very simple, depending only upon the principles of approximation, explained Art. 50.

PLATE III.

72. Take, as an example, to find the tangents at the origin and the direction of flexure of the corresponding branches of the curve, whose equation is

$$x^2(y-b)=y^2(x-a),\quad a>b.$$

The first approximation gives

$$ay^2-bx^2=0.$$

The equations of the tangents are therefore

$$y\sqrt{a}=\pm x\sqrt{b}.$$

The second approximation for the branch, whose tangent is $y\sqrt{a}=x\sqrt{b}$, is found from

$$y\sqrt{a}-x\sqrt{b}+\frac{xy(x-y)}{y\sqrt{a}+x\sqrt{b}}=0,$$

in which, neglecting powers of x above the square,

$$y\sqrt{a}-x\sqrt{b}+\frac{\sqrt{(ab)}(\sqrt{a}-\sqrt{b})}{2\sqrt{(ab)}.a}x^2=0;$$

therefore y is less in the branch than in the tangent for the same value of x.

Changing the sign of $\sqrt{b}$, the form of the other branch is given by

$$y\sqrt{a}+x\sqrt{b}+\frac{\sqrt{a}+\sqrt{b}}{2a}x^2=0.$$

The directions of the flexure in both branches is given in the figure. Fig. 6.

73. We may observe that the curve

$$v_1+Ax^2=0$$

is a parabola whose axis is parallel to the axis of y.

It is easy to shew, that an approximate parabolic form may be found, having an axis parallel to any given line $mx+ny=0$, for $\frac{u_3}{w_1}$ may be written $\frac{u_3}{w_1(mx+ny)^2}(mx+ny)^2$, and a second approximation to the form is

$$v_1+\frac{f(b,-a)}{(a'b-b'a)(mb-na)^2}(mx+ny)^2=0,$$

which is the parabola required.

PLATE III.

74. Instead of using the property of ratios which has been applied, we might obtain the next approximation to the curve, $v_1 w_1 + u_3 + \ldots = 0$, by writing in $\frac{u_3}{w_1}$ the value of y obtained from $v_1 = 0$, in accordance with the principles of approximation given in the last chapter. This may be considered generally the easier method to practice.

Take for examples the forms of the following curves near the origin:

(1) $$a^2(x^2 - y^2) + 2axy^2 + ay^3 - x^4 - x^2y^2 = 0.$$

The tangents at the origin are $y = \pm x$.

For the branch whose tangent is $y = x$, since

$$y = x + \frac{2xy^2 + y^3}{a(y+x)} + \ldots,$$

the second approximation is $y = x + \dfrac{3x^2}{2a}$  .

For the branch whose tangent is $y = -x$, since

$$y = -x + \frac{2xy^2 + y^3}{a(y-x)} + \ldots,$$

the second approximation is $y = -x - \dfrac{x^2}{2a}$  .

Both branches bend towards the negative end of the axis of x.

(2) $$x^4 - y^4 - a^2x^2 + b^2y^2 = 0.$$

For the direction of flexure from $ax = by$,

$$by - ax = \frac{y^4 - x^4}{by + ax} = \frac{a^4 - b^4}{2ab^4}\, x^3.$$

For the flexure from $-ax = by$,

$$by + ax = \frac{y^4 - x^4}{by - ax} = -\frac{a^4 - b^4}{2ab^4}\, x^3.$$

Plate II., fig. 15, is the form of such a curve for particular values of a and b.

PLATE III.

(3) $$y^4 - 2a^2y^2 + 2a^2x^2 - 3ax^3 + x^4 = 0,$$

near the origin, $$y^2 = x^2 - \frac{3x^3}{2a};$$

$$\therefore\ y = \pm x\left(1 - \frac{3x}{4a}\right).$$

The curve is symmetrical with respect to Ox, and since

$$y^2(2a^2 - y^2) = x^2(x-a)(x-2a),$$

$$\text{near } (a, 0),\ 2a^2y^2 = -a^3\xi,$$

$$\text{near } (2a, 0),\ 2a^2y^2 = \ a^3\xi,$$

$$\text{near } (0, a\sqrt{2}),\ \text{if } y = a\sqrt{2} + \eta$$

$$-2a^2.2\sqrt{2}a\eta = 2a^2x^2,$$

$$\text{near } (a, a\sqrt{2}),\ -2a^2.2\sqrt{2}a\eta = -a^3\xi,$$

$$\text{near } (2a, a\sqrt{2}),\ -2a^2.2\sqrt{2}a\eta = \ 4a^3\xi.$$

To find where the curve is parallel to the axis of x, let (α, β) be such a point, we must observe that if $x = \alpha + \xi$ and $y = \beta + \eta$, the resulting equation in ξ, η represents a curve passing through the origin and touching the axis of ξ, so that η vanishes compared with ξ near the point of contact.

The substitution in the proposed equation gives, near (α, β),

$$(4a^2\alpha - 9a\alpha^2 + 4\alpha^3)\,\xi + P\eta + Q\xi^2 + \ldots = 0,$$

since $$\beta^4 - 2a^2\beta^2 + 2a^2\alpha^2 - 3a\alpha^3 + \alpha^4 = 0;$$

therefore, making ξ very small,

$$4a^2\alpha - 9a\alpha^2 + 4\alpha^3 = 0,$$

whence $\alpha = 0$ and $\frac{1}{8}\{9 \pm \sqrt{(17)}\}\,a$, or $\frac{13}{8}a$ and $\frac{5}{8}a$ nearly; these values give the positions of the points required, viz. a, c, b and b', symmetrically placed on opposite sides of Ox.

Similarly, the curve is parallel to Oy, where

$$4y^3 - 4a^2y = 0,\ \text{ or } y = 0 \text{ and } y = \pm a,$$

Fig. 7. when $$y = a,\ x^4 - 3ax^3 + 2a^2x^2 - a^4 = 0.$$

The values of x are the abscissæ of the points of intersection of

$$x^2 = ay \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and $$y^2 - 3xy + 2ay - a^2 = 0,$$

or $$y(y - 3x + 2a) = a^2 \ \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

PLATE III.

If the parabola (1) and the hyperbola (2) be constructed, it will be seen that, since the parabola cuts the asymptote $y-3x+2a=0$ at the points $x=a$ and $2a$, the points of intersection, p and q, have for their abscissæ a small negative value and a positive value a little greater then $2a$, and that these are the only points of intersection. Fig. 8.

The points so determined are the points e, e', and f, f', in the given curve. Fig. 7.

The student may also find the solutions of the biquadratic by the intersections of $ay=x^2$ and $y(x-a)(x-2a)=a^3$.

75. If there be no functions of the third order, the equation of the curve being $v_1w_1+u_4+\ldots=0$, the next approximation may be obtained from

$$v_1+\frac{u_4}{w_1x^3}\cdot x^3=0,$$

$$\therefore\ v_1+Bx^3=0,$$

which expresses that the distance of a point of the corresponding branch from the tangent changes sign as we pass through the point of contact, or that there is a point of inflexion.

76. Take, as an example, the curve

$$a^2(x^2-y^2)+x^4+y^4=0.$$

The branch, whose tangent is $y=x$, has the approximate form of

$$y-x+\frac{2x^4}{2a^2x}=0, \text{ or } y=x+\frac{x^3}{a^2}.$$

The curve is symmetrical with respect to both axes, and meets the axis of y where $y=\pm a$.

Near $(0, a)$, let $y=a+\eta$, and write the equation in the form

$$(a^2+x^2)x^2+y^2(y^2-a^2)=0;$$

$$\therefore\ x^2+2a\eta=0,$$

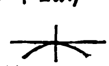
giving the parabolic form  .

PLATE III.

Or, a form of the second approximation may be written $x^2+\eta^2+2a\eta=0$, which shews that the radius of curvature is a.

Fig. 9. The dotted line in the figure is the circle of curvature at the points $(0, \pm a)$.

$$u_3 \equiv v_1 v_2.$$

77. If v_1 be a factor of u_3, the equation being

$$v_1 w_1 + v_1 v_2 + u_4 + \ldots = 0,$$

the form of the branch, whose tangent is $v_1 = 0$, is given by

$$v_1 + \frac{u_4}{w_1 + v_2} = 0,$$

and therefore by $v_1 + \dfrac{u_4}{w_1} = 0$, or $v_1 + Ax^3 = 0$,

hence, this branch has a point of inflexion.

In this case, v_1 being of the third order, $v_1 v_2$ is of the fifth, and therefore ranks after u_4.

78. The following curves will serve as examples:

(1) $a^2(x+2y)(y-2x) - a(y-2x)x^2 + y^4 = 0.$

For the direction of flexure from the tangent $y - 2x = 0$, the next approximation, obtained from

$$y - 2x + \frac{y^4}{a^2(x+2y)} = 0,$$

by making $y = 2x$ in the smaller term, is

$$y - 2x + \frac{16x^3}{5a^2} = 0.$$

The second approximation to the branch, whose tangent is $x + 2y = 0$, is $x + 2y - \dfrac{x^2}{a} = 0$.

The curve cuts the axis of x where $x = a$; near this point, if $x = a + \xi$, since the equation becomes, when y^2 is rejected,

$$-3a^2xy - ax^2y + 2ax^2(x-a) = 0,$$

PLATE III.

the approximate equation is $-2y+\xi=0$. The parabolic asymptote is $2ax^3+y^4=0$. To give some idea of the size of the loop, which joins the ends of the branches in xOy,

$$x=y=\frac{\pm\sqrt{(13)}-1}{2}\,a=1.3a,\text{ or }-2.3a\text{ nearly.}$$

Both branches are above the asymptote, crossing it a considerable distance. Fig. 10.

The calculation of the direction of flexure of the branches at the multiple point might have been avoided, as remarked in Art. 55, by observing that $y-2x=0$ does not meet the curve except at the origin, consequently there are three consecutive points in the tangent, and therefore a point of inflexion; and that $x+2y=0$ meets it again at the point $(-40a, 20a)$, too far to be represented in the figure.

(2) $$a^3(y^2-x^2)-2a^2(y^3+x^3)+ay^4+x^5=0.$$

The tangents to the branches through the origin are $y=\pm x$.

For the upper sign $a(y-x)-2x^2=0$,

.... lower ..... $a^2(y+x)-\frac{1}{2}x^3=0$.

The equation may be written in the form

$$y^2(y-a)^2=x^2(x+a)(a^2+ax-x^2),$$

or $a\{(y-\frac{1}{2}a)^2-\frac{1}{4}a^2\}^2=x^2(x+a)(x+2a\sin 18^\circ)(2a\sin 54^\circ-x)$.

The curve is therefore symmetrical with respect to the line $y=\frac{1}{2}a$, and y is impossible, if $x>2a\sin 54^\circ$, or between $-2a\sin 18^\circ$ and $-a$. The forms, where $x=-a$, $-2a\sin 18^\circ$, and $2a\sin 54^\circ$, are common parabolas.

The infinite branch is of the form of $x^5+a(y-\frac{1}{2}a)^4=0$. Fig. 11.

We may illustrate the method of Art. 45 in approximating to the position of the points of intersection of the curve with the line $y=\frac{1}{2}a$, given by the equation

$$x^5-2a^2x^3-a^3x^2+\frac{a^5}{16}=0,$$

the values of x in this equation are the abscissæ of the points of intersection of the curves

$$a^2y=x^3,\text{ and }x^2(y-a)-2a^2y+\frac{a^3}{16}=0,$$

PLATE III. the asymptotes of the latter are

$$y = a, \text{ and } x = \pm a\sqrt{2}.$$

The first curve is the dotted curve, the asymptotes of the second are omitted.

The letters m, n, r determine the position of M, N, R in

Fig. 12. the figure of the curve.

II. $u_2 \equiv v_1^2$.

79. The second case, when $u_2 \equiv v_1^2$, in which there are two coincident tangents, gives, generally, as the approximate form $v_1^2 + u_3 = 0$; hence, if A be the value of $\frac{u_3}{x^3}$ when $b, -a$ are written for x and y, $v_1^2 + Ax^3 = 0$; therefore if A be finite, *i.e.* unless v_1 be a factor of u_3,

$$ax + by = \pm(-Ax^3)^{\frac{1}{2}};$$

so that, generally, when u_3 exists, as in Art. 17, this curve has a ceratoid cusp at the origin.

80. An example of this form occurs in the curve

$$(ax - by)^2 - ax^2y - y^4 = 0.$$

Near the origin,

$$(ax - by)^2 - \frac{a^2}{b}x^3 = 0, \text{ or } by = ax \pm \frac{a}{\sqrt{b}}x^{\frac{3}{2}};$$

near the points, $x = 0$, $y = \pm b$, let $y = \pm b + \eta$,

then $\qquad \pm 2ab^2x \mp b^2.2b\eta = 0$, or $b\eta + ax = 0$;

$$y = a, \; x = \infty, \text{ or } -\frac{a^2 - b^2}{2b}.$$

If x and y be both very large, $ax^2 + y^3 = 0$, which meets the

Fig. 13. curve where $x = -\frac{b^3}{a^2}$. The figure is drawn for $a > b$.

PLATE III.

$$u_3 \equiv v_1 v_2.$$

81. When v_1 is a factor of u_3, we have to obtain the approximate form from $v_1^2 + v_1 v_2 + u_4 = 0$; in which case

$$(v_1 + \tfrac{1}{2}v_2)^2 = \tfrac{1}{4}v_2^2 - u_4, \text{ or } v_1 = -\tfrac{1}{2}v_2 \pm Bx^2 = (C \pm B)\,x^2.$$

If B be impossible, the origin is a conjugate point.

If B be real and different from zero, each of the branches is parabolic, the curvatures being different.

If $B = 0$, or $4u_4 = v_2^2$, the form must be found from the next approximation, viz. from

$$(v_1 + \tfrac{1}{2}v_2)^2 + u_5 = 0,$$

therefore $$v_1 + \tfrac{1}{2}v_2 = Dx^{\frac{5}{2}},$$

and, as in Art. 17, the form is seen to be a ramphoid cusp.

82. The following examples will illustrate these forms:

(1) $(x-y)^2 - (x-y)\,x^2 - \tfrac{1}{3}x^4 - \tfrac{1}{9}y^4 = 0,$

near the origin $x - y = \tfrac{4}{3}x^2$, or $-\tfrac{1}{3}x^2$,

$$y = 0,\ x^2 - x^3 - \tfrac{1}{3}x^4 = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

whence $$x = \tfrac{5}{8} \text{ and } -3\tfrac{5}{8} \text{ nearly};$$

near the points where $x = 0$, $y = \pm 3$, let $y = \pm 3 + \eta$; it can then be shewn that

$$\eta = -x + \tfrac{1}{6}x^2 \text{ and } \eta = -x + \tfrac{5}{6}x^2,$$

so that the curve bends upwards at both the points where the curve meets the axis of x, which for the upper point is contrary to what one would have expected without calculation.

The tangent $y + x = 3$, it will be found, meets the curve at points where $x = 1\tfrac{5}{8}$ and $-2\tfrac{7}{16}$ nearly.

To obtain a good idea of the size of the curve, observe that $x = -y = \tfrac{3}{2}$ or -6. We can shew that near these points the forms are given by $3\eta + 13\xi = 0$, and $3\eta = 7\xi$.

The directions of the curve at the points where the curve intersects the axis of x, are deduced from a curious property of the curve which may be shewn as follows: Fig. 14.

PLATE III.

Let the curve be cut by a line $x+y=a$, the lines joining the origin with the four points of intersection are given by the homogeneous equation

$$(x^2-y^2)^2 - ax^2(x^2-y^2) - \tfrac{1}{3}a^2x^4 - \tfrac{1}{3}a^2y^4 = 0,$$

or $$(1-a-\tfrac{1}{3}a^2)\,x^4 - (2-a)\,x^2y^2 + (1-\tfrac{1}{3}a^2)\,y^4 = 0.\ldots \text{(ii)}.$$

When all four points are real, these lines form two pairs, equally inclined to the axes, if two only are real, the two lines are equally inclined to the axes.

Hence, a pair are coincident at each of the points of intersection with the axis of x; this appears directly from equation (ii), since $y^4=0$ when $1-a-\frac{1}{3}a^2=0$, which makes $y+x=a$ satisfied by the points given in (i).

The same property shews that there is a double tangent parallel to $x+y=0$, touching at a, b, so that Oa, Ob, are equally inclined to Oy.

(2). $$2\,(x-y-\tfrac{2}{3}x^2)^2 + (x+y)\,y^4 - y^6 - x^6 = 0,$$

near the origin

$$(x-y-\tfrac{2}{3}x^2)^2 + x^5 = 0, \text{ or } y = x - \tfrac{2}{3}x^2 - (-x)^{\frac{5}{2}},$$

a ramphoid cusp.

$$y=0,\ \ x^2=0 \text{ and } 2\,(1-\tfrac{2}{3}x)^2 - x^4 = 0,$$

$$x^2 \pm \tfrac{2}{3}\sqrt{2}x + \tfrac{2}{9} = \tfrac{2}{9} \pm \sqrt{2} = \tfrac{1}{9}\,(2 \pm 13) \text{ nearly},$$

$$\therefore\ x + \tfrac{1}{3}\sqrt{2} = \pm\tfrac{1}{3}\sqrt{(15)}, \text{ or } x = \tfrac{23}{27} \text{ and } -\tfrac{49}{27} \text{ nearly};$$

$$\text{where } x=0,\ 2y^2 + y^5 - y^6 = 0,$$

$$\therefore\ y^2=0 \text{ and } y^4 - y^3 - 2 = 0, \text{ or } y = -1 \text{ or } \tfrac{3}{2} \text{ nearly},$$

$$\text{where } x=y,\ \tfrac{8}{9}x^4 + 2x^5 - 2x^6 = 0,$$

$$x^2 - x + \tfrac{1}{4} = \tfrac{1}{4} + \tfrac{4}{9} = \tfrac{25}{36},\ x = \tfrac{4}{3} \text{ or } -\tfrac{1}{3},$$

Fig. 15. $$x = -y = \tfrac{9}{8} \text{ or } -\tfrac{43}{24} \text{ nearly}.$$

III. $u_2 \equiv v_1^2 + w_1^2$.

83. The third case, when u_2 is of the form $v_1^2 + w_1^2$ gives a conjugate point, since the first approximation has $v_1=0$, $w_1=0$, as the only solutions.

PLATE III.

84. Take the following curve as an example:

$$a^2(x^2+y^2)-2a(x-y)^3+x^4+y^4=0.$$

There is a conjugate point at the origin.

To trace the curve, observe that $-x$ for y and $-y$ for x do not alter the equation, hence the curve is symmetrical with respect to the line bisecting xOy',

$$x=0,\quad y=-a \text{ two values,}$$

$$y=0,\quad x=a \quad \text{............}$$

If $x=-y$, $a^2-8ax+x^2=0$, $x=(8-\frac{1}{8})a$ or $\frac{1}{8}a$ nearly,

near $(0,-a)$ $a^2\eta^2-6a^3\xi=0$, 

near $(a,0)$ $a^2\xi^2+6a^3\eta=0$, 

$y=mx$ meets the curve where

$$a^2(m^2+1)+2a(m-1)^3x+(m^4+1)x^2=0;$$

therefore if x be real

$$(m^4+1)(m^2+1)-(m-1)^6<0;$$

$$\therefore\ 6m^5-14m^4+20m^3-14m^2+6m<0,$$

or $$2m\{3(m^2+1)-4m\}(m^2+1-m)<0;$$

therefore m must be negative, hence, except in the angle xOy', there is no part of the curve, which consists of an oval and the conjugate point. Fig. 16.

CURVATURE OF BRANCHES AT MULTIPLE POINTS.

85. To find the diameter of curvature of the branch of the curve whose tangent is $u_1=0$, the equation of the curve being $u_1v_1+u_3+u_4+\ldots=0$.

Let $\quad u_1\equiv ax+by,\quad v_1\equiv a'x+b'y,\quad u_3\equiv f(x,y)$.

The second approximation gives $u_1v_1+u_3=0$.

The diameter of curvature is the limit of $\pm\dfrac{x^2+y^2}{u_1}\sqrt{(a^2+b^2)}$,

PLATE III.

or the limit of $\mp \frac{(x^2+y^2)\,v_1}{u_2} \sqrt{(a^2+b^2)}$, when $\frac{x}{b} = \frac{y}{-a}$, it is therefore $\mp \frac{a'b-b'a}{f(b,-a)} (a^2+b^2)^{\frac{3}{2}}$.

86. To compare the diameters of curvature of the branches through the origin of the curve

$$(a-x)\,y^2 = (4a-y)\,x^2.$$

The equation may be written

$$a\,(y-2x)\,(y+2x) - xy\,(y-x) = 0.$$

The diameters of curvature to the two branches are the limits of $\frac{x^2+y^2}{y \mp 2x} \cdot \sqrt{5}$, or $\frac{(x^2+y^2)(y \pm 2x)}{xy\,(y-x)} \sqrt{5}\,.\,a$, when $y = \pm 2x$; they are $5\sqrt{5}\,.\,\frac{1}{2}a$ and $5\sqrt{5}\,.\,\frac{1}{6}a$, in the ratio 3 : 1.

MULTIPLE POINTS OF HIGHER ORDERS.

87. These discussions of particular forms of the homogeneous functions are sufficient to shew how they can be dealt with in all ordinary cases.

The different forms which u_3 could assume, if the first of the existing functions were of the third degree, will suggest themselves readily, as when there is only one real branch, when there are three branches, when two of the three or all three osculate; also the variations which arise in consequence of peculiar forms of the succeeding functions similar to those mentioned in Art. 70.

Thus, there would be three osculating branches, if the equation reduced to the form

$$(u_1 - t_2)\,(u_1 - v_2)\,(u_1 - w_2) + u_7 + \ldots = 0.$$

88. A few of the varieties are exhibited in the following curves.

(1) $$a\,(y^2 - x^2)\,(y - 2x) - y^4 = 0.$$

This curve has three branches through the origin, the

PLATE III.

deflexions from the tangents to which are easily seen from the approximate equations

$$a(y-x)+\tfrac{1}{2}y^2=0,$$
$$a(y+x)-\tfrac{1}{8}y^2=0,$$
$$a(y-2x)-\tfrac{1}{3}y^2=0,$$
$$\text{near } (0, a),\ -2a^3x-a^3(y-a)=0.$$

The infinite branch is $2ax^3=y^4$. Fig. 17.

(2) $$a(y-x)(x^2+y^2)+x^4=0.$$

Through the origin there is a branch of the form of

$$2a(y-x)+x^2=0,$$

there is also a conjugate point called a point-circle,

$$x^2+y^2=0,$$
$$\text{near } (a, 0),\ y+x-a=0,$$
$$\text{near } (\infty, \infty),\ ay^3+x^4=0.$$ Fig. 18.

(3) $$a(y-x)^2(y+x)-y^4-x^4=0,$$

near the origin $$a(y-x)^2-x^3=0,$$

and $$2a(y+x)-x^2=0,$$

and x and y being interchanged does not alter the equation, therefore there is symmetry with respect to the line $x=y$,

$$\text{near } (a, 0),\ -a^3y-a^3(x-a)=0.$$ Fig. 19.

(4) $$a^2(y-x)^3-2ax^4+ay^4-x^5=0,$$

$$\text{near the origin } a^2(y-x)^3-ax^4=0,$$

or $$y-x=\left(\frac{x^4}{a}\right)^{\frac{1}{3}},$$

$$\text{near } (0, -a),\ -a^4\eta-3a^4x=0, \text{ if } y=-a+\eta;$$
$$\text{near } (-a, 0),\ 3a^4y+a^3\xi^2=0, \text{ if } x=-a+\xi,$$
$$\text{near } (\infty, \infty),\ ay^4-x^5=0.$$ Fig. 20.

(5) $$2a^2(y-x)^2(y+x)-4ax^3(y-x)+2x^5-x^4y=0.$$

For the branches whose common tangent is $y=x$,

$$4a^2(y-x)^2-4ax^2(y-x)+x^4=0$$

PLATE III.

is the next approximate equation which gives two coincident parabolas

$$2a(y-x)-x^2=0.$$

To separate the branches we must conduct the approximation a step further;

$$\therefore\ 4a^2(y-x)^2\left(x+\frac{x^2}{4a}\right)-4ax^3(y-x)+x^5-\frac{x^6}{2a}=0;$$

$$\therefore\ \{2a(y-x)-x^2\}^2+\frac{x^5}{4a}-\frac{x^5}{2a}=0;$$

$$\therefore\ 2a(y-x)=x^2+\left(\frac{x^5}{4a}\right)^{\frac{1}{2}},$$

which gives a ramphoid cusp.

For the branch whose tangent is $y+x=0$,

$$a(y+x)+x^2=0,$$

$$y=0,\ 2a^2x^3+4ax^4+2x^5=0,$$

$$\text{or } x^3(x+a)^2=0,$$

near $(-a, 0)$, $\quad -2a^2x^2y-4ax^3y-x^4y+2x^3(x+a)^2=0,$

$$\therefore\ (x+a)^2=\tfrac{1}{2}ay.$$

It will be seen that the asymptotes are

Fig. 21. $\quad y=2x-4a,\ 2a^2y^2=x^4.$

IV.

(1) $\quad ax^2(y-a)=y^3(x-2a).$

Find the tangent and the direction of flexure at the point $(2a, a)$.

(2) $\quad (y-2)^2=(x-1)^2x.$

Find the branches which pass through $(1, 2)$, and find the radius of curvature at the point where the curve meets the axis of y.

(3) $\quad (2a-x)y^2=x(a+b-x)^2,\ b<a.$

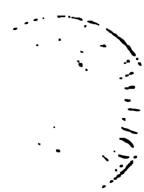
Find the tangent to the curve at the point $(a+b, 0)$.

Trace the curve when $2b=a$,

PLATE III.

(4) Trace the curve

$$x^4 - 2\sqrt{2}\,ax^3 + 2a^2x^2 - a^2y^2 - ay^3 = 0.$$



(5) $$y^4 + 2x(x - 3a)\,y^2 - 2ax^3 + a^2x^2 = 0.$$

Find the shape at the origin, and where $2x = a$.

(6) Trace the two curves

$$x^4 - ax^2y + \tfrac{1}{4}a^2y^2 - ay^3 = 0,$$

$$\text{and } x^4 - ax^2y - \tfrac{3}{4}a^2y^2 - ay^3 = 0.$$

(7) $$(a - x)\,y^2 = (a + x)\,x^2.$$

Shew that the centres of the circles of curvature of the branches through the origin are at the points $(-a, \pm a)$.

Shew that the greatest breadth of the loop measured parallel to Oy is $(\sqrt{5} - 1)\,a$.

(8) $$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}.$$

Prove that the radius of curvature at the point where $x = y$ is $\frac{3}{2}c$.

(9) $$x^4 + y^4 - 2ax^3 + a^2(x + y)^2 = 0.$$

Shew that there is a ceratoid cusp at the origin, and find the form of the curve near $(a, 0)$.

(10) In the last curve, shew that only one tangent, besides the axis of x, passes through the origin and that m, the tangent of the inclination to the axis of y is given by the equation

$$m^5 + 2m^4 + m^3 + m + 2 = 0.$$

Construct for its value by the method of Art. 45.

(11) $$x^6 + y^6 - a^2(x^2 - y^2)^2 = 0.$$

Find the forms of the branches which pass through the origin, and shew that the radius of curvature of each branch is a.

(12) $$y^3(y - x) = a(x^3 + y^3).$$

Find the form of the curve near the origin, and at the point $(0, a)$.

CHAPTER V.

FORMS OF BRANCHES WHOSE TANGENTS AT THE ORIGIN ARE THE COORDINATE AXES.

PLATE III.

89. I have now to call attention to the cases in which x and y are of different orders of magnitude, for points taken very near the origin along any branch of a curve whose equation is given.

Such cases arise when the equation of the curve can be shewn to be correctly replaced by a simpler equation of the form $ax^m = by^n$, when m and n are unequal.

For example, if the equation of a curve were

$$x^4 + ax^2y + bxy^2 + cy^3 = 0.$$

The simpler equation $x^2 + ay = 0$ would make bxy^2 of the order x^5, and cy^3 of the order x^6, and the equation would be $x^2 + ay(1 + \epsilon) = 0$, where by diminishing x, ϵ might be made as small as we please; so that one series of values of x and y will satisfy the equation $x^2 + ay = 0$, which, therefore, gives the form of one branch of the curve.

90. The variety of cases which arise is so great, that I shall attempt no subdivision of the two cases in which x is small compared with y, and y small compared with x, reserving for a future chapter the application of Newton's parallelogram and De Gua's analytical triangle to the discrimination of the branches of a curve which pass through the origin; it will there be seen that these artifices save the trouble of thinking in a great measure, but my object is rather to give the student distinct ideas of the work, which he is doing in making approximations, than merely to obtain results.

PLATE III.

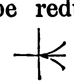
91. I shall therefore in this chapter discuss such a number of particular cases as will be sufficient to shew how any case, which is not very complicated, may be handled, taking as examples curves, which are represented by equations which either contain only a small number of terms, or are reducible, by simple considerations, to a small number of terms.

92. A simple case to begin with is the curve whose equation is

$$x^4 - axy^2 + y^4 = 0.$$

Here x and y cannot be of the same order of magnitude, since in that case axy^2 would be of the order x^3, and the equation could not hold near the origin.

We try first whether x can be small compared with y, in which case x^4 would be small compared with y^4, and the equation would become $-axy^2 + y^4 = 0$, or $ax = y^2$; therefore, since for a short distance along this curve, whose form is , the value of x^4, which we neglected, is of the order y^8, this is an approximate form of one part of the curve.

We then try whether y can be small compared with x, in which case y^4 could be neglected compared with x^4, and the equation would be reduced to $x^3 - ay^2 = 0$; along this curve, whose form is , y^4 is of the order x^6, and the result is consistent with the assumption that y^4 might be rejected. This is therefore another branch at the origin.

The curve is then easily drawn, considering that it is symmetrical with respect to Ox, and that x cannot be negative. Fig. 22.

93. The next which I shall consider is a more difficult case, and will serve as an illustration of the circumstance that, if the rationalized equations of a curve be arranged in homogeneous functions of x and y ascending in degree, such as $u_r + u_s + ... = 0$, u_r does not necessarily belong to the terms which can give the first approximation to every branch of

PLATE III.

the curve passing through the origin; that is, the functions are not arranged in order of magnitude for all branches of the curve.

The equation of the curve is

$$x^3y^2 + xy^5 - y^7 - x^7 = 0.$$

Here x and y cannot be of the same order of magnitude; we must try, therefore, whether $x : y$ or $y : x$ can be small.

First, suppose $y : x$ small, *i.e.* as small as we please by diminishing x; if so, y^7 is small compared with x^7 and the equation reduces so far to

$$x^2y^2 + y^5 - x^6 = 0.$$

Again, $\qquad y^5 : x^2y^2 = y^3 : x^2,$

which is as small as we please; therefore y^5 may be neglected and $y^2 = x^4$ gives an approximate form,  where we observe that the terms neglected are of the orders x^{11} and x^{14} while those retained are of the order x^7.

Next, suppose $x : y$ small, in which case x^7 vanishes compared with y^7, and the equation is reduced to

$$x^3 + xy^3 - y^5 = 0.$$

If in any branch x^3 can be neglected $x = y^2$, and $x^3 = y^6$, hence the term neglected is small compared with y^5 retained,  is therefore the form of a branch.

If xy^3 could be neglected, $x^3 = y^5$, then one condition would be satisfied, viz. that $x : y$ should be small, but $xy^3 \propto y^{\frac{5}{3}+3}$ or $y^{4+\frac{2}{3}}$, which is greater than the term y^5, retained; there is therefore no such branch through the origin.

If y^5 be neglected, $x^2 = -y^3$, and y^5, which $\propto x^{\frac{10}{3}}$, is small compared with the term x^3 retained,  is therefore the form of a branch.

The four branches have thus been shewn to be given by the equations

$$y = x^2, \quad y = -x^2, \quad x = y^2, \quad \text{and} \quad x^2 = -y^3.$$

The terms of the original equation, which give rise to the branch $x=y^2$, are xy^5-y^7; therefore the term x^3y^2, which is the first term in the arrangement by homogeneous functions, is of the order y^8 in this branch, and does not form an element in the equation giving the approximation.

It will hereafter be seen how, by placing the terms of this equation in De Gua's triangle, the trouble of this discussion is removed, but I have thought it better to shew by this method how to select the combinations which give the forms of the branches, although when the number of terms is large, the examination becomes too complicated for practice, and recourse must be had to the triangle.

94. When there are only three terms in the equation, as in $x^3+y^3-3axy=0$, it is easy to obtain the form at the origin by trying whether, on neglecting any term, the resulting relation makes that term small compared with those retained.

Thus neglecting x^3, $y^2=3ax$, with which relation x^3 is of the order y^6, therefore properly neglected.

Similarly, neglecting y^3, $x^2=3ay$, and y^3 is of the order x^6.

But if xy be neglected, $y+x=0$; therefore xy is of the order x^2, and could only be neglected upon supposition of x and y being very great.

95. If these tentative methods should be attempted in cases of equations which contain a large number of terms, the number is capable of being greatly reduced by the following considerations:

(1) The coefficient of any power of y being a function of x, we can reject, as small by comparison, all but the term involving the lowest power of x in that coefficient.

(2) Similarly for the coefficient of any power of x.

Moreover, say that we are going to try for a branch in which y is small compared with x, in this case,

PLATE III.

(3) from any homogeneous function of more than one term, which may form part of the equation, we may reject all the terms except that which contains the highest power of x.

(4) If, after the above simplifications, a term x^n remain not having y in its coefficient, no term involving y need be retained, in which the sum of the indices of x and y is n or any greater quantity, and no other term involving x will appear.

Hence in the residual equation, to which the tentative method is to be applied, the indices of x must descend by steps of 2 at least, although the indices of y may ascend by steps of 1.

Thus, if x^6 be the highest power of x, the residual equation might be

$$x^6 + ax^4y + bx^2y^2 + cy^3 = 0, \text{ or } x^6 + ax^4y + by^4 = 0.$$

96. The student should obtain the branches near the origin in the following curves, and after this practice he will be probably able to perform the requisite operations with ease, the arguments passing through his mind almost involuntarily.

(1) $$ax(y-x)^2 - y^4 = 0.$$

Near the origin, x and y of the same order gives $y = x \pm \left(\frac{x^3}{a}\right)^{\frac{1}{2}}$; $x : y$ small gives $ax = y^2$, $y : x$ small gives $ax^3 = y^4$, which contradicts the assumption if x and y are small, but agrees with it if x and y be very large; hence

Fig. 23. $ax^3 = y^4$ gives the direction of bending at an infinite distance.

(2) $$x^4 - a^2xy - b^2y^2 = 0.$$

Near the origin, $a^2x + b^2y = -\frac{b^2x^3}{a^2}$, and $x^3 - a^2y = 0$,

near (∞, ∞), $x^4 = b^2y^2$,

Fig. 24. and the curve is symmetrical in opposite quadrants.

PLATE III.

(3) $$x^5 - a^2(x^3 + y^3) + a^3xy = 0.$$

Near the origin, x^5 may be rejected, and the branches

are $x^2 = ay$, and $y^2 = ax$,

near (∞, ∞), $x^5 = a^2y^3$,

near $(\pm a, 0)$), $2\xi \pm \eta = 0$. Fig. 25.

(4) $$x^5 - 3bx^3y - bxy^3 + 4b^2y^3 = 0.$$

Near the origin, xy^3 may be neglected;

$\therefore\ x^2 = 3by$, and $3x^3 = 4by^2$,

near (∞, ∞), y^3 may be neglected, therefore $x^4 = by^3$;

also, when $x = 4b$, $y = \infty$ or $\frac{16}{3}b$,

when $x = y$, $x^2 - 4bx + 4b^2 = 0$,

when $x = 2b$, $y = 2b$ or $-4b$.

There is a multiple point at $(2b, 2b)$, the direction of whose branches are given by

$$11\xi^2 - 12\xi\eta + 3\eta^2 = 0.$$ Fig. 26.

(5) $$x^5 - ax^3y - axy^3 + a^2y^3 = 0.$$

Near the origin, $x^2 = ay$ and $x^3 = ay^2$;

near (∞, ∞), $x^4 = ay^3$,

also, when $x = a$, $y = \infty$ or a,

when $x = y$, $(x - a)^2 = 0$.

The curve is parallel to the axis of y where

$$27(x - a)x + 4a^2 = 0.$$ Fig. 27.

(6) $$(y^2 + x^2)^2 - 6axy^2 - 2ax^3 + a^2x^2 = 0.$$

This equation may be written

$$(y^2 + x^2 - 3ax)^2 = 4ax^2(2a - x).$$

There are, therefore, two osculating branches through the origin given by

$$y^2 = (3 \pm \sqrt{8})\,ax,$$

x cannot be negative nor greater than $2a$;

near $(2a, a\sqrt{2})$, if $x = 2a + \xi$, $y = a\sqrt{2} + \eta$, $\eta^2 + 2a\xi = 0$,

near $(a, 0)$, $-4a^2y^2 + a^2\xi^2 = 0$,

and the curve is symmetrical with respect to Ox. Fig. 28.

PLATE III.

(7) $$y^4 - 2axy^2 - 3a^2x^2 + x^4 = 0.$$

There are two osculating branches at the origin,

$$y^2 = 3ax \text{ and } -ax;$$

$$\text{near } (\pm a\sqrt{3}, 0), \text{ if } x = \pm a\sqrt{3} + \xi, \; y^2 = 3a\xi;$$

$$\text{also } (y^2 - ax)^2 = x^2(4a^2 - x^2);$$

therefore $x = 2a$ is a tangent where $y = \pm a\sqrt{2}$.

Shew that the curve is parallel to Ox at the points whose

Fig. 29. coordinates are $1{\cdot}6a$, $1{\cdot}87a$ and $-1{\cdot}08a$, $\cdot 86a$.

PLATE IV.

(8) $$x^5 - ax^3y - 2a^2xy^2 + a^2y^3 = 0.$$

Near the origin, if $y : x$ be small,

$$x^4 - ax^2y - 2a^2y^2 = 0,$$

$$\therefore x^2 = -ay \text{ or } 2ay, \text{ and } y^3 \text{ is of the order } x^6;$$

x and y of the same order of magnitude gives

$$y = 2x + \frac{x^2}{2a};$$

$$\text{near } (\infty, \infty), \; x^5 + a^2y^3 = 0, \text{ near } (2a, 4a), \; \eta + 7\xi^2 = 0.$$

The curve really crosses the asymptote at $(8a, -2a)$, too

Fig. 1. far to be shewn in the figure.

(9) $$x^6 + ax^4y - cx^2y^2 + dxy^3 \pm ey^4 = 0.$$

If x and y be of the same order of magnitude

$$-cx^2 + dxy \pm ey^2 = 0$$

gives two ordinary branches through the origin if the roots be real.

If y be small compared with x,

$$x^4 + ax^2y - cy^2 = 0$$

Fig. 2. gives two osculating parabolic branches.

If the lower sign be taken

$$x^6 - cy^4 = 0$$

gives two semi-cubical parabolic asymptotes, since with this

Figs. 3 & 4. relation the remaining terms are of orders lower than x^6.

Fig

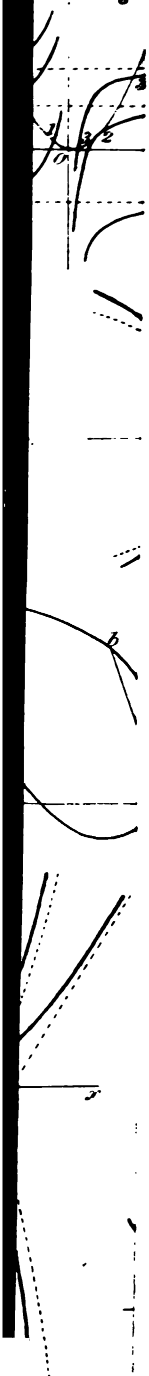


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PLATE IV.

(10) $$x^6 + a^2y^4 + a^2x^3y + a^3xy^2 = 0.$$

Near the origin, $x : y$ small, $y^2 + ax = 0$,

$$y : x \text{ small}, \; x^5 + a^2x^2y + a^3y^2 = 0,$$

$$\therefore \; x^3 + a^2 y = 0, \text{ and } x^2 + ay = 0;$$

there is therefore a triple point. Fig. 5.

97. The following examples will serve for practice of the method spoken of in Art. 94, in which the equations of the curves either contain only three terms, or are reducible immediately to three terms by the considerations given in Art. 95.

The curves which I have selected can be traced completely by the methods already described, and will be useful as an introduction to the chapters on asymptotes which succeed.

(1) $$x^4 - axy^2 - a^3y = 0.$$

Rejecting xy^2 compared with y, $x^4 = a^3y$ is the form at the origin; rejecting x^4, $yx + a^2 = 0$ is an approximation, if x be finite, y infinite, therefore the axis of y is an asymptote, and the curve lies to the left above, to the right below; lastly, rejecting y, $x^3 = ay^2$ holds if x and y be both infinite, a semi-cubical parabolic asymptote. Fig. 6.

(2) $$x^4 - axy^2 - ay^3 = 0.$$

Neglecting x^4, $x + y = 0$, near the origin; neglecting y^3, $x^3 = ay^2$ makes y^3 of the order $x^{4+\frac{1}{2}}$, and gives a semi-cubical parabola at the origin; neglecting xy^2, $x^4 = ay^3$ makes xy^2 of the order $x^{3+\frac{2}{3}}$, correctly neglected at an infinite distance. Fig. 7.

(3) $$x^4 - a^2xy - ay^3 = 0.$$

$$\text{Near } (0, 0), \; ax + y^2 = 0 \text{ and } x^3 - a^2y = 0;$$

$$\text{near } (\infty, \infty), \; x^4 = ay^3.$$ Fig. 8.

(4) $$x^4 + ax^2y - ay^3 = 0.$$

$$\text{Near } (0, 0), \; x^2 = y^2 \text{ and } x^2 + ay = 0;$$

$$\text{near } (\infty, \infty), \; x^4 = ay^3.$$ Fig. 9.

PLATE IV.

(5) $$x^4 - axy^2 - a^2y^2 + by^3 = 0.$$

Near the origin, axy^2 and by^3 may be rejected compared with a^2y^2, so that $x^4 - a^2y^2 = 0$ gives two branches through the origin.

Near (∞, ∞), y^2 vanishes compared with y^3;

$$\therefore x^4 - axy^2 + by^3 = 0.$$

If x^4 be neglected, $ax = by$; therefore x^4 was improperly neglected compared with the terms kept.

If xy^2 be neglected, $x^4 + by^3 = 0$, and xy^2, being of the order $x^{1+\frac{8}{3}}$, was properly neglected.

If y^3 were neglected, $x^3 - ay^2 = 0$, y^3 would be of the order $x^{4+\frac{1}{2}}$, or greater than the terms retained; therefore $x^4 + by^3 = 0$ is the only asymptote. The figure is drawn for

Fig. 10. $a = 2b$ nearly.

(6) $$x^5 - a^3xy - a^2y^3 = 0.$$

$$\text{Near } (0, 0),\ ax + y^2 = 0 \text{ and } x^4 - a^3y = 0;$$

Fig. 11. $$\text{near } (\infty, \infty),\ x^5 - a^2y^3 = 0.$$

(7) $$x^5 - a^3xy - ay^4 = 0.$$

$$\text{Near } (0, 0),\ a^2x + y^3 = 0 \text{ and } x^4 - a^3y = 0;$$

Fig. 12. $$\text{near } (\infty, \infty),\ x^5 - ay^4 = 0.$$

(8) $$x^3y^2 - ax^4 + a^2y^3 = 0.$$

$$\text{Near } (0, 0),\ x^4 = ay^3;$$

Fig. 13. $$\text{near } (\infty, \infty),\ x^3 + ay = 0, \text{ and } y^2 - ax = 0.$$

(9) $$x^3y^2 - a^3x^2 + ay^4 = 0.$$

$$\text{Near } (0, 0),\ a^2x^2 = y^4;$$

$$\text{near } (\infty, \infty),\ x^3 + ay^2 = 0;$$

Fig. 14. $$\text{near } (\infty, 0),\ xy^2 - a^3 = 0.$$

(10) $$x^5 - ax^4 - a^3xy + ay^4 = 0.$$

$$\text{Near the origin, } x^4 + a^2xy - y^4 = 0,$$

$$\text{rejecting } x^4,\ a^2x - y^3 = 0,$$

$$\ldots\ldots\ldots\ y^4,\ x^3 + a^2y = 0;$$

PLATE IV.

near (∞, ∞), $x^5 + ay^4 = 0$;

$x = a$, $y = 0$ or a,

near $(a, 0)$, $y = x - a$,

...... (a, a), $4\xi^2 + 3\eta = 0$.

Fig. 15.

V.

Trace the curves:

(1) $(x^2+y^2)^2 = a^2xy$.

(2) $y^4 + 2x(x-3a)y^2 - 2ax^3 + a^2x^2$.

(3) $(x^2+y^2)^2 - 6axy^2 - 2ax^3 + 2a^2x^2 = 0$.

(4) Shew that the asymptote of the curve, $ax(y-x)^2 - y^4 = 0$, cuts the curve at a finite distance at a point at which the curve runs parallel to the axis of y. Find at what point the curve is parallel to the axis of x.

(5) Draw the two curves $x^4 - 2ax^2y \pm a^2y^2 - ay^3 = 0$.

(6) Find the points at which the curve $x^4 - axy^2 - ay^3 = 0$ is parallel to the axes.

Draw the following curves:

(7) $x^4 - a^2xy + by^3 = 0$.

(8) $x^4 - 4ax^2y + 4a^2y^2 - ax^3 = 0$.

(9) $a^2xy - 2ax^2y + x^4 + y^4 = 0$.

(10) $x^5 - ax^3y + by^4 = 0$.

(11) $x^5 - ax^2y^2 + by^4 = 0$.

(12) $x^5 - axy^3 + by^4 = 0$.

(13) $x^5 - a^2x^2y + by^4 = 0$.

(14) $x^5 - a^2xy^2 + by^4 = 0$.

(15) $y^5 + ax^4 = b^2xy^2$.

(16) $x^5 - ay^4 + 2bx^3y + b^2xy^2 = 0$.

(17) $a^2y^3 - a^2by^2 + a^2x^2y + x^5 = 0$.

(18) $(x^2+y^2)^3 = 4a^2x^2y^2$.

(19) $y^6 + x^6 = x^2(y-2x)(y+x)$.

PLATE IV.

I must first remind the student that, if $ax^n + bx^{n-1} + \ldots + k = 0$ be an equation which has one infinite root, then $a = 0$; if it have two infinite roots, both $a = 0$ and $b = 0$, and so on.

This is easily seen if for x we write $\frac{1}{z}$, which makes the equation $a + bz + \ldots + kz^n = 0$; and this equation must be satisfied by $z^2 = 0$, if there be two roots equal to zero.

102. Some elementary treatises on Algebraical Geometry speak of a straight line, drawn parallel to the axis of the parabola, as meeting the curve in one point; this, although justifiable from one point of view, is not a satisfactory way of disposing of the question concerning the intersection of a parabola with a straight line, and is liable to mislead the student in his examination of curves in general, and especially with respect to the branches which are at an infinite distance.

The separation of curves into classes of different degrees is effected by arranging them analytically into curves in which the highest powers of the current coordinates are of the 1^{st}, 2^{nd}, $\ldots n^{th}$ degrees; or, geometrically, into curves which are met by a straight line in 1, 2, ... n points.

There are obvious advantages, in choosing an interpretation of the analytical and geometrical classification, which makes the classes coextensive, and it is unfortunate when expressions are used which are inconsistent with such an interpretation.

To take the case of a parabola, alluded to above, as an example,

$$y^2 = 4ax$$

may be considered as a particular case of the general equation of the second degree, from which some of the terms are excluded, which ought to have appeared in the form

$$0.x^2,\ 0.xy,\ \&c.,$$

in this point of view the straight line $y = c$, parallel to the axis, intersects the curve at points whose abscissæ are given by the equation

$$0.x^2 + 4ax - c^2 = 0,$$

one root of which is infinitely great and the other finite.

Thus the geometrical and analytical arrangements coincide.

And this does not interfere with the geometrical definition of a parabola as a curve which is the locus of a point whose distances from a fixed point and a fixed straight line are equal; for a point which is at an infinite distance satisfies this condition, and is therefore a point in the curve.

On this principle, if constant $= 0$ be considered as a particular case of an equation of the first degree, it is the equation of a straight line at an infinite distance; if it be considered as a particular case of the equation of the second degree, it is the equation of a conic section or two straight lines at an infinite distance.

103. In the intersection of a curve of the n^{th} degree by any straight line, we ought to be able to account for all the n points of intersection, and, if the equations of the curve and straight line give a resulting equation of the $(n-r)^{\text{th}}$ degree, we must conclude that r points of intersection, real or imaginary, are at an infinite distance.

Now, as no straight line can be drawn, which is so near a curve in the neighbourhood of any point, as the particular line which passes through the next consecutive point as well, Art. 59, the same being true, however distant these points are, it follows that the line which has two points at an infinite distance is *generally* the nearest line that can be drawn to the curve at an infinite distance, being the rectilinear asymptote.

104. In order to find what kind of exceptions may occur to this general statement, we ought to consider the variations which occur in finding the tangent to a curve at a point at a finite distance, and so deduce the corresponding variations in cases of rectilinear asymptotes.

105. In the first place, at a point of inflexion where the straight line is constructed which passes through the point of inflexion and a consecutive point, it, at the same time,

PLATE IV.

passes through a second consecutive point on the opposite side; so that, although we can only generally construct a straight line so as to pass through two points, it happens in this particular case to pass through a third. The corresponding case for an asymptote, is that of three points at an infinite distance, an example of which has been given in Art. 3.

106. In the next place, if we try to find a tangent to a branch of a multiple point, say a double point, we observe that any straight line which passes through the multiple point, passes through two points in the curve, one for each branch, in whatever direction it be drawn; and that having fulfilled this condition, after satisfying which, one constant in the equation is still undetermined, the second constant enables us to make the line pass through another point on either branch, so that when this point is taken indefinitely near to the multiple point, the straight line becomes a tangent to the branch on which it lies.

107. Suppose now this multiple point to be at an infinite distance, if any straight line be drawn through it, all straight lines parallel to this straight line will also pass through the same point; thus, if $y - mx = 0$ be a line passing through such a point, any line $y = mx + \alpha$ will also pass through it; hence, if $y = mx + \alpha$ be combined with the equation of the curve, there will be r infinite roots of the resulting equation in x or y, if the multiple point have r branches, whatever be the value of α. The rectilinear asymptote, which corresponds to the tangents to a branch, must then be found by the condition that one more root has an infinite value; and, if the condition determines more than one such asymptote, each of these are the tangents to the branches of the multiple point supposed at an infinite distance.

108. Take as an example the curve of the third degree, whose equation is

$$x(y-x)^2 - b^2y = 0,$$

PLATE IV.

any straight line $y = x + a$ gives an equation

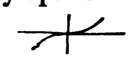
$$0.x^3 + 0.x^2 + (a^2 - b^2)\,x - b^2 a = 0,$$

in which case there are two infinite values of x for all lines parallel to $y = x$. In order to obtain the nearest possible straight line to the curve at an infinite distance, we must make $a^2 = b^2$, so that either of the straight lines, $y = x \pm b$, meets the curve at three points at an infinite distance.

When $x = 0$, the values of y are given by

$$0.y^3 + 0y^2 - b^2 y = 0,$$

which shews the axis of y is an asymptote.

Near the origin $x^3 - b^2 y = 0$, ; also the curve is symmetrical in opposite quadrants. Fig. 16.

109. In this curve we observe that there are two branches at an infinite distance from both axes, and that these two branches both pass through the point in which all lines parallel to $y = x$ intersect, so that this point is a true multiple point at an infinite distance; and each of these parallel lines intersects the curve at two points at an infinite distance, for that reason. $y = x + b$ represents a straight line which passes through two consecutive points on the same branch, and the third point at an infinite distance is that in which it meets the branch whose tangent is $y = x - b$.

110. These considerations of infinitely distant points of intersection supply a complete method of determining rectilinear asymptotes to a curve, as will be seen by a few examples. It is well to know this method, although that of approximation has a decided advantage, both because it supplies a knowledge of the side of the asymptote on which the curve lies, and also because it gives the curvilinear as well as the rectilinear asymptotes.

111. The general method spoken of may be explained by the following process.

PLATE IV.

Consider the intersection of a curve of the n^{th} degree with a straight line whose equation is $y = \alpha x + \beta$.

If we eliminate y we obtain an equation of the n^{th} degree in x, giving the n values of x at the points of intersection, real or imaginary; this equation is of the form

$$Px^n + Qx^{n-1} + Rx^{n-2} + \ldots = 0,$$

where P is a function of α, and Q, R, &c. are functions of α and β.

Since two values of x must be infinite, we have the equations $P = 0$ and $Q = 0$, which generally determine α and β.

Thus if the terms of the n^{th} degree in the equation of the curve be

$$ay^n + bxy^{n-1} + \ldots + lx^n,$$

$$P \equiv a\alpha^n + b\alpha^{n-1} + \ldots + l,$$

and $P = 0$ determines n values of α, real or imaginary.

If α_1 be one of these values, we have learned that the straight line $y = \alpha_1 x + \beta$ meets the curve at one point at an infinite distance; if now we substitute this value of α in the equation $Q = 0$, β can be determined, and since Q is of the first degree in β, there is only one such value of β for each value of α.

Thus, generally, there are n asymptotes completely determined by these equations.

112. The exceptions to this general statement are of great variety; it will be sufficient to take one case, viz. that in which $P = 0$ has two equal roots, each equal to α_1, in which case, when $\alpha = \alpha_1$

$$na\alpha^{n-1} + (n-1)\,b\alpha^{n-2} + \ldots = 0, \text{ Art. } 38,$$

that is, the coefficient of β in Q vanishes for this value of α, and there will be no corresponding rectilinear asymptote unless Q vanishes independently of the value of β. This is the case of a multiple point at an infinite distance, of which an example has been already given, Art. 108.

In this case the equation $R=0$ determines the two values of β, which correspond to the tangents to the two branches of the multiple point. PLATE IV.

113. This exceptional case may be further illustrated by the curve

$$y(y-x)^2(y-2x)+3a(y-x)x^2-2a^2x^2=0.$$

The equation may also be written in the form

$$(y-x)^4-(y-x-2a)(y-x-a)x^2=0.$$

Each of the lines $y-x=2a$, and $y-x=a$, meets the curve at four points at an infinite distance, three of which are consecutive points on the infinite branch to which each is a tangent, and the fourth on the branch to which the other is a tangent.

This is a case of a multiple point of two branches at an infinite distance, each having a point of inflexion at the common point.

It is easily seen that no point of the curve lies between these two asymptotes.

Again $y-2x=\beta$ meets the curve where

$$0.x^4+(2x+\beta)(x+\beta)^2\beta+3a(x+\beta)x^2-2a^2x^2=0,$$

and a second point is at an infinite distance, if $2\beta+3a=0$; therefore $y-2x+\frac{3}{2}a=0$ is an asymptote, which meets the curve at points when $9y^2=17x^2$.

The two points at a finite distance are given by

$$\tfrac{19}{4}x^2-\tfrac{27}{2}ax+\tfrac{81}{16}a^2=0,$$

whence $\frac{9}{4}a=\{3\pm\frac{1}{2}\sqrt{(17)}\}x$, and $x=\frac{9}{20}a$ or $\frac{9}{4}a$ nearly.

A fourth asymptote $2y+3a=0$ meets the curve at two points infinitely distant, and two at a finite distance.

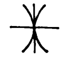
The shape at the origin is $y^4=2a^2x^2$, ; and, at $(-\frac{2}{3}a,\ 0)$, the tangent is $y=\frac{9}{13}(x+\frac{2}{3}a)$. Fig. 17.

PLATE IV.

where it meets the asymptote,

$$(-m+1)x'' + \tfrac{1}{2}(a+b) = 0;$$

$$\therefore\ -x' + x'' = \frac{am^3+b}{m(1-m^2)} - \frac{a+b}{2(1-m)}$$

$$= \frac{a(2m^3-m^2-m) - b(m^2+m-2)}{2m(1-m^2)}$$

$$= \frac{-ma(1+2m)+b(2+m)}{2m(m+1)}.$$

Hence the curve is further from the origin than the asymptote or nearer according as $am(2m+1) <$ or $> b(2+m)$ and they intersect, if $ra = b$, when

$$m^2 + \tfrac{1}{2}(1-r)m - r = 0,$$

whose roots are real if r be positive, and since, with the proportions given in the figure, $r = \frac{1}{3}$ nearly,

$$m^2 + \tfrac{1}{3}m - \tfrac{1}{3} = 0,\ \therefore\ m = \tfrac{13}{30} \text{ and } -\tfrac{23}{30} \text{ nearly,}$$

when m is nearly 1, the difference of the distances from the origin is $\frac{3}{4}(a-b)\sqrt{2}$.

This ultimate constant distance does not prevent the indefinite approach of the curve to the asymptote, since the perpendicular distance is ultimately $\frac{3}{4}(a-b)\sqrt{2} \times \sin\theta$, where θ is indefinitely small.

(4) $$y^2(x^2-y^2) - 2ay^3 + 2a^3x = 0.$$

$x \mp y = 0$ gives one infinite root, $x \mp y = \alpha$ meets the curve where $0.y^4 + y^2\alpha(\alpha \pm 2y) - 2ay^3 + 2a^3(\alpha \pm y) = 0$, two roots are infinite, if $\pm\alpha = a$, and the finite roots are given by

$$a^2y^2 \pm 2a^3y \pm 2a^4 = 0, \text{ or } (y \pm a)^2 = (1 \mp 2)a^2,$$

the roots are impossible for the upper sign, and $(1 \pm \sqrt{3})a$ for the lower.

$y = 0$ gives three infinite roots, and $y^2 = -\dfrac{2a^3}{x}$ nearly; therefore the curve lies on both sides of the negative end of xOx'; near the origin, $y^3 = a^2x$; near $(0, -2a)$, $4\eta + x = 0$. Fig. 21.

DETERMINATION OF ASYMPTOTES BY APPROXIMATION.

116. I shall now shew in this and following chapters how the asymptotes of a curve may be found by successive approximations, and also how the side of the asymptote on which the curve comes into sight from an infinite distance may be discovered.

117. In order to arrive at all the infinite branches of a curve, we must examine the following cases which include all possible ways in which a curve can pass off to infinity.

(1) x may be infinite, while y is finite or 0.

(2) y may be infinite, while x is finite or 0.

Both x and y may be infinite, dividing into three cases.

(3) x and y may be of the same order of magnitude, or $x : y$ finite.

(4) x may be large compared with y, or $y : x$ vanish ultimately, when x and y are increased indefinitely.

(5) y may be large compared with x, or $x : y$ vanish ultimately, when x and y are increased indefinitely.

Classes (1) and (2) include the cases in which curves run off to infinity parallel to the axes of coordinates.

Class (3) includes rectilinear asymptotes which are inclined at finite angles to the axes, and as special cases parabolic asymptotes.

Classes (4) and (5) include general curvilinear asymptotes.

These will be discussed separately.

CASE OF x ALONE INFINITE.

118. To try whether x can be infinite while y remains finite, we must suppose the equation arranged in descending powers of x in the form

$$f(y)\,x^r + \phi(y)\,x^s + \ldots = 0,$$

$$\text{or } f(y) + \phi(y)\,\frac{1}{x^{r-s}} + \ldots = 0,$$

hence making x infinite, $f(y) = 0$.

PLATE IV.

In order, therefore, that this case should occur, it is necessary that the term containing the highest power of x in the equation should involve y, and, if b be a root of $f(y)=0$, $y=b$ is generally an asymptote, since, even if there be not two equal roots, the condition of having two roots at an infinite distance is satisfied, because, if the curve be of the n^{th} degree, r cannot be greater than $n-1$, nor s greater than $n-2$, and in the equation resulting from the substitution of b for y the first finite term can involve no higher power than x^{n-2}. If $f(y)=0$ have t roots, real or imaginary, r cannot be greater than $n-t$, nor s greater than $n-t-1$; therefore, when $y=b$, in the resulting equation the first finite term cannot involve a higher power of x than x^{n-t-1}, hence these must be $t+1$ infinite roots.

The $t+1$ infinite roots are accounted for as follows: the equation $f(y)=0$ gives t parallel lines, which therefore all meet at an infinite distance in the same point; any straight line parallel to these lines meets the curve in t points at an infinite distance, but is not an asymptote, since it does not contain on itself two such points; each of the parallel lines corresponding to solutions of $f(y)=0$ contains two such points besides the $t-1$ in which it intersects the other lines.

This is the case of a multiple point of t branches at an infinite distance, spoken of in Art. 107.

119. To find the side of the asymptotes on which the curve lies we must proceed to the next approximation.

If $y=b$ be a single root of $f(y)=0$, so that

$$f(y)=(y-b)\,f_1(y), \text{ where } f_1(b) \text{ is not zero},$$

$$y-b+\frac{\phi(y)}{f_1(y)}\,\frac{1}{x^{r-s}}+\ldots=0;$$

$$\therefore \text{ if } \phi(b) \text{ be not zero, } y=b-\frac{\phi(b)}{f_1(b)}\,\frac{1}{x^{r-s}}.$$

Hence, when $r-s$ is even, the curve is on the same side of the asymptote $y=b$ at both ends, above or below according as $\dfrac{\phi(b)}{f_1(b)}$ is negative or positive.

PLATE IV.

When $r-s$ is odd, the curve lies above the asymptote at one end and below at the other.

If
$$\phi(b)=0,$$
and
$$y-b+\frac{\phi(y)}{f_1(y)}\frac{1}{x^{r-s}}+\frac{\psi(y)}{f_1(y)}\frac{1}{x^{r-t}}+\ldots=0,$$
the next approximation gives
$$y-b+\frac{\psi(b)}{f_1(b)}\frac{1}{x^{r-t}}=0.$$

If $f(y)=0$ have two roots $=b$, so that
$$f(y)=(y-b)^2 f_2(y),\ \text{ and } f_2(b) \text{ be not zero,}$$
$$(y-b)^2+\frac{\phi(b)}{f_2(b)}\frac{1}{x^{r-s}}=0,$$
is the next approximation.

When $r-s$ is even, if $\dfrac{\phi(b)}{f_2(b)}$ be negative, the curve lies on both sides of the asymptotes at each end, if $\dfrac{\phi(b)}{f_2(b)}$ be positive, the asymptotes are imaginary; when $r-s$ is odd, the curve lies on both sides at the positive end, if $\dfrac{\phi(b)}{f_2(b)}$ be negative, and at the negative end, if it be positive.

120. If the equation be supposed to be arranged in descending powers of x, for the sake of determining whether x can be infinite while y is not so, difficult cases of determining the direction of infinite branches arise, when there are many equal values which make the coefficients of the first term vanish, and the same value makes the coefficients of following powers of x also vanish. In such cases it often happens that the term involving the highest power of x is not one of the principal terms for all the infinite branches running parallel to Ox.

Thus, in the case of the curve
$$(y-b)^4x^4+ay^3(y-b)x^3+a^7x+a^8=0.$$

The second term will give a correct approximation to the value of y, viz. $b-\dfrac{a^6}{b^3x^2}$, because with this value $(y-b)^4x^4$ is

PLATE IV.

of the order $\frac{1}{x^4}$, and the terms retained are of the order x.

The first term, however, is one of the principal terms for another branch, thus taking the first and second terms together,

$$(y-b)^3 + \frac{ab^3}{x} = 0,$$

which relation makes the terms retained of the order $x^{\frac{5}{3}}$ and therefore larger than those rejected.

121. The following examples of varieties which may occur will sufficiently explain the application of these processes; the asymptotes which are not parallel to the axes may be taken as additional illustrations of the method by infinite roots given above.

(1) $$(x-a)\,y^2 - a^2x = 0.$$

$x = a$, and $y^2 = a^2$ are asymptotes

$$\text{near } (a, \infty),\ x - a = \frac{a^2x}{y^2} = \frac{a^3}{y^2},$$

$$\ldots\ldots\ (\infty, a),\ y - a = \frac{ay^2}{(y+a)\,x} = \frac{a^2}{2x}.$$

These determine the sides on which the curve lies, when it comes into sight from infinity in the direction of these asymptotes.

To trace the curve, we observe that it is symmetrical with respect to the axis of x, and that, near the origin,

Fig. 22. $y^2 + ax = 0$.

(2) $$(x-a)(x-b)\,y^2 - a^2x^2 = 0,\quad a > b.$$

$x = a$ and b, $y^2 = a^2$ are four asymptotes;

$$\text{near } (a, \infty),\ x - a = \frac{a^2x^2}{(x-b)\,y^2} = \frac{a^4}{(a-b)\,y^2},$$

$$\ldots\ldots\ (b, \infty),\ x - b = \frac{a^2x^2}{(x-a)\,y^2} = -\frac{a^2b^2}{(a-b)\,y^2},$$

$$\ldots\ldots\ (\infty, a),\ \{x^2 - (a+b)\,x\}\,y^2 - a^2x^2 = 0,$$

$$\text{or } y^2 - a^2 = (a+b)\,\frac{y^2}{x},\ \text{ therefore } y - a = (a+b)\,\frac{a}{2x}.$$

PLATE IV. Fig. 23.

There is symmetry with respect to Ox, and near the origin, $by^2 = ax^2$.

The asymptotes $y^2 = a^2$ meet the curve where $(a+b)\,x = ab$. No part of the curve lies between the other asymptotes.

Note 1. When $a = b$, the curve degenerates into two hyperbolas $(x-a)\,(y \mp a) = \pm a^2$, whose form is obvious from the figure when the asymptotes move up to one another.

Note 2. When $b = 0$, the curve degenerates into the axis of y, combined with the locus in the last example.

(3) $\quad y^2x\,(y-x) - ay^3 - byx^2 + a\,(a+b)\,x^2 = 0.$

The equation may also be arranged thus,

$$y^3\,(x-a) - x^2\,(y-a)\,(y+a+b) = 0.$$

The asymptotes parallel to the axes are

$$x = a,\quad y = a \text{ and } -(a+b)\,;$$

$$\text{near } (a, \infty),\quad x - a = \frac{a^2}{y},$$

$$\ldots\ldots\ (\infty, a),\quad y - a = \frac{a^3}{(2a+b)\,x},$$

$$\ldots\ldots\ \{\infty, -(a+b)\},\quad y + a + b = \frac{(a+b)^3}{(2a+b)\,x}.$$

If $y - x = \alpha$, the points of intersection are given by

$$0.x^4 + (\alpha - a - b)\,x^3 + \ldots = 0,$$

and $y - x = a + b$ is therefore an asymptote;

$$\text{near the origin, } y^3 = (a+b)\,x^2,$$

$$\text{near } (a, a),\quad a\xi = (2a+b)\,\eta,$$

$$\ldots\ldots\ \{a, -(a+b)\},\quad (a+b)^3\xi = a^2\,(2a+b)\,\eta.$$

The figure is drawn for $b = a$ nearly. Fig. 24.

Note 1. The curve is parallel to Oy, where

$$y = -b - \sqrt{(3a^2 + 3ab + b^2)}\,;$$

the other root gives impossible values of x; in the figure $y = -\frac{18}{6}a$ nearly.

Note 2. The curve is parallel to Ox where $x = 2a$. There are two positive and one negative value of y.

PLATE IV.

(4) $$(x-a)\,y^3=(y-b)^2\,x^2.$$

The asymptotes parallel to the axes are

$$x=a \text{ and } (y-b)^2=0,$$

$$\text{near } (a,\infty),\ x-a=\frac{a^2}{y},$$

$$\ldots\ldots\ (\infty,b),\ (y-b)^2=\frac{b^3}{x}.$$

Rearranging the equation

$$xy^2(y-x)-ay^3+2bx^2y-b^2x^2=0,$$

$y-x=\beta$ meets the curve at points given by the equation,

$$0.x^4+(\beta-a+2b)\,x^3+\ldots=0,$$

therefore at two points at infinity, if $\beta=a-2b$, or $y-x=a-2b$ is an asymptote, which meets the curve in the points given by

$$(3b^2-a^2)\,x^2-2\,(a+b)\,(2b-a)^2\,x+a\,(2b-a)^3=0,$$

the roots of which are real, if $2b>a$.

$$\text{Near } (0,0),\ ay^3+b^2x^2=0,$$

$$\ldots\ldots\ (a,b),\ b^3\xi-a^2\eta^2=0.$$

The curves are drawn one for $a<b\sqrt{3}$, the other for $a>2b$. Figs. 25, 26.

Note. The curve is parallel to Ox, where $x=2a$, and to Oy where $y=3b$, unless these two take place simultaneously, which will occur when the point $(2a, 3b)$ is a point on the curve, in which case $16a=27b$.

With regard tot his multiple point, we may observe that, if $(\alpha+\xi, \beta+\eta)$ be a point near (α,β), the tangent at such a point is parallel to Ox if $\eta=A\xi^2$, and to Oy if $\xi=B\eta^2$, but there is a multiple point if $A\xi^2+2B\xi\eta+C\eta^2=0$.

In the first case the coefficient of ξ in the expansion vanishes, in the second that of η, and in the third the coefficients of both ξ and η vanish,

In the case considered, $2a+\xi$, $3b+\eta$ substituted in the equation gives

Fig. 27. $$27^2\eta^2=3.16^2\xi^2, \text{ or } \eta=\pm\sqrt{\left(\tfrac{256}{243}\right)}\,\xi=\pm\tfrac{38}{37}\,\xi \text{ nearly.}$$

It is easily seen how the form of the curve for this case separates the two forms drawn for $a < \sqrt{3}\, b$ and $> 2b$. PLATE IV.

(5) $$(x^2 - a^2)(x - 2a)\, y^2 - a^2 x^3 + 4a^4 y = 0.$$

There are three asymptotes parallel to Oy,

$$x = \pm a \text{ and } 2a;$$

also two parallel to Ox, viz. $y^2 = a^2$

$$\text{near} \quad (a, \infty), \quad x = a + \frac{2a^2}{y},$$

$$\ldots\ldots (-a, \infty), \quad x = -a - \frac{2a^2}{3y},$$

$$\ldots\ldots (2a, \infty), \quad x = 2a - \frac{4a^2}{3y},$$

$$\ldots\ldots (\infty, a), \quad y = a + \frac{a^2}{x},$$

$$\ldots\ldots (\infty, -a), \quad y = -a - \frac{a^2}{x},$$

$$x = \pm a, \quad y = \pm \tfrac{1}{4}a,$$

$$x = 2a, \quad y = 2a,$$

$$y = a, \quad x = \tfrac{3}{2}a \text{ or } -2a,$$

$$y = -a, \; x \text{ impossible.}$$

Near $(0, 0)$, $x^3 = 4a^2 y$,

$\ldots\ldots$ $(0, -2a)$, $x + \eta = 0$,

$\ldots\ldots$ $(2a, 2a)$, $2a\eta + 5\xi^2 = 0$,

$$x = \tfrac{3}{2}a, \quad y = \tfrac{27}{5}a.$$ Fig. 28.

(6) $$(xy - ab)^2 = b^2 d\,(c - y).$$

Both axes are asymptotes,

$$\text{near} \quad (0, \infty), \quad x = \pm b \sqrt{\left(-\frac{d}{y}\right)},$$

$$\ldots\ldots (\infty, 0), \quad y = b.\frac{a \pm \sqrt{(cd)}}{x},$$

$$x = 0, \quad y = c - \frac{a^2}{d}.$$

PLATE IV.

Figs. 29, 30.

The first figure is drawn for $a^2 > cd$, the second for $a^2 < cd$.

Note. The curve is parallel to Oy where

$$2x(xy - ab) = -b^2 d,$$

whence $$(2cx^2 - ab)^2 = b^2(a^2 - cd),$$

the roots of which are only possible when $a^2 > cd$.

The distance between the tangents parallel to Oy in the first figure is $\frac{b}{c}\sqrt{(a^2 - cd)}$.

$y = c$ gives equal values of x, where the curve runs parallel to Ox, in both figures.

(7) $$x^3y^2 - 2a^2x^2y + a^4x - b^5 = 0,$$
$$\text{or } x(xy - a^2)^2 - b^5 = 0.$$

The axes are both asymptotes,

$$\text{near } (0, \infty),\ x^3 = \frac{b^5}{y^2},$$
$$\ldots\ldots (\infty, 0),\ xy = a^2 - \left(\frac{b^5}{x}\right)^{\frac{1}{2}},$$

therefore x cannot be negative; and, on the positive side, $y = \frac{a^2}{x}(1 \pm \alpha)$, where α vanishes when x is indefinitely great; hence the curve comes in sight from an infinite distance like the branches of a ramphoid cusp.

$$y = 0,\ a^4x = b^5,$$

and the curve is parallel to Ox, where

Fig. 31. $$3xy = a^2,\ 4a^4x = 9b^5.$$

(8) $$x^2y^2 - 2a^2xy - b^2y^2 + 2a^2b^2 - b^4 = 0.$$

The asymptotes are $x^2 = b^2$ and $y^2 = 0$,

$$\text{near } (\pm b, \infty),\ x \mp b - \frac{a^2}{y} = 0,$$
$$\ldots\ldots (\infty, 0),\ (xy - a^2)^2 = (a^2 - b^2)^2,$$
$$\therefore\ y = \frac{b^2}{x},\ \text{and } \frac{2a^2 - b^2}{x}.$$

The asymptotes $x = \pm b$ intersect the curve, where PLATE IV.

$$\pm 2a^2 y = (2a^2 - b^2) b;$$

$$\text{where } x = 0,\ y^2 = 2a^2 - b^2,$$

$$y = \pm b,\ \pm bx = a^2 \pm \sqrt{\{b^4 + (a^2 - b^2)^2\}}.$$

The curve is symmetrical in opposite quadrants.

The figures are drawn for the cases in which $2a^2 >$ and $< b^2$; in the critical case in which $2a^2 = b^2$ the curve is compounded of the axis of x and another curve whose equation is $(x^2 - b^2)\, y = b^2 x$. Figs. 32, 33.

VI.

(1) $$y (x - a)^2 = x (y - 4a)^2.$$

Trace the curve, and shew that the radii of curvature of the branches through $(a, 4a)$, are $5^{\frac{3}{2}} a$.

(2) $$(ay - x^2)^2 = y^2 (b^2 - x^2).$$

Trace the curve when $a <$, and when $> b$.

Find where the curve runs parallel to Oy.

(3) $$(x - a)(x + a)^2 y^2 - a^2 x^3 + 4a^4 y = 0.$$

Find the sides of the asymptotes on which the curve lies, and trace the curve.

(4) $$a^2 (y - x)^2 = x^2 (y^2 + x^2).$$

Trace the curve, and shew that any straight line, drawn perpendicular to the tangent at the origin, intersects the curve in points from which the radii to the origin are, in pairs, equally inclined to the axes.

(5) $$x^2 (x^2 - y^2)^2 = a^2 (x^2 + y^2).$$

Find the asymptotes parallel to the axis of y, and from these and the points of intersection of the line $x - y = \alpha$ with the curve, for different values of α, deduce the form of the curve.

PLATE IV.

(6) $$(x-a)\,y^4 + x^2y^3 \mp b^5 = 0.$$

Trace the two curves, and shew that in the first there is one, and in the second three, points, at which the curve is parallel to the axis of x.

(7) $$x^4y^3 = a^2\,(a-x)^3\,(a-2x)^2.$$

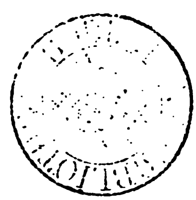
(8) $$x^2y^3 - a^3xy + a^5 = 0.$$

Trace the last two curves, and in the last find the points at which the curve is parallel to the axes.

(9) $$4xy^3 = 3\sqrt{3}\,a^2\,(x^2+y^2-a^2).$$

Trace the curve shewing that there is a multiple point at

$$\left(\frac{1}{\sqrt{2}}a,\ \sqrt{\frac{3}{2}}a\right).$$



CHAPTER VII.

ASYMPTOTES NOT PARALLEL TO THE AXES. ASYMPTOTES TO HOMOGENEOUS CURVES.

PLATE V.

122. In this chapter I shall shew how to approximate to the forms of the infinite branches which are not ultimately parallel to either axes.

In the first place, I shall consider the cases in which x and y, being both infinite, can be of the same order of magnitude. This is the general case of rectilinear asymptotes, although by including rectilinear asymptotes which are at an infinite distance, certain parabolic asymptotes appear in the course of the examination.

In the second part of the chapter I shall add some more illustrations, to those which have already been incidentally given, of the cases in which x and y are of different orders of magnitude.

123. In order to make the general statement more easy to understand, it will be well to take a few particular cases, and to shew how the method of approximation is to be employed to determine the asymptotes, and the side on which the curve lies, when it first comes into sight from an infinite distance.

(1) $$x^2(y-b) = y^2(x-a).$$

Arranging the equation in homogeneous functions descending in degree,

$$xy(y-x) - bx^2 + ay^2 = 0.$$

For the first approximation, neglecting quantities of the second order, compared with those of the third, $y = x$.

PLATE V.

The second approximation is obtained from

$$y - x - \frac{bx}{y} + \frac{ay}{x} = 0,$$

by writing $y = x$ in the smaller terms, so that

$$y - x - b + a = 0$$

is the equation of the asymptote.

For the third approximation, we must write, according to Art. 50, the value of y obtained from the second approximation, whence

$$y - x - b\left(1 + \frac{b-a}{x}\right)^{-1} + a\left(1 + \frac{b-a}{x}\right) = 0,$$

and neglecting terms of higher orders than $\frac{1}{x}$,

$$y - x - b\left(1 - \frac{b-a}{x}\right) + a\left(1 + \frac{b-a}{x}\right) = 0,$$

$$\therefore\ y = x + b - a - \frac{b^2 - a^2}{x};$$

therefore, if $b > a$, the curve lies below the asymptote at the positive and above it at the negative end. Fig. 1.

(2) $$x^3 + y^3 - 3axy = 0.$$

First approximation $y + x = 0$, and since

$$y + x - \frac{3axy}{x^2 - xy + y^2} = 0,$$

writing $-x$ for y in the smaller term, the second approximation gives

$$y + x + a = 0.$$

For the side of the asymptote on which the curve lies, since $x^2 - xy + y^2 = a^2 - 3xy$,

$$y + x + a\left(1 - \frac{a^2}{3xy}\right)^{-1} = 0,$$

whence $$y + x + a - \frac{a^3}{3x^2} = 0,$$

whence the curve lies above the asymptote at both ends.

PLATE V.

Another method of determining the side on which the curve lies is by assuming as the next approximation $y+x+a=\alpha$, where α is small compared with a.

Substituting in the original equation,

$$x^3-(x+a-\alpha)^3+3ax(x+a-\alpha)=0,$$

$$3\alpha x^2-a^3=0, \text{ neglecting } \alpha^2,$$

which gives $$y+x+a=\frac{a^3}{3x^2}.$$ Fig. 2.

(3) $$(x-a)y^3=(y-2a)^2x^2.$$

First approximation, $y=x$, and since

$$xy^3-y^2x^2-ay^3+4ax^2y-4a^2x^2=0,$$

$$y-x-a\frac{y}{x}+4a.\frac{x}{y}=0,$$

$$\therefore\; y-x-a+4a=0$$

is the asymptote.

For the next approximation to the form of the curve

$$y-x-a\left(1-\frac{3a}{x}\right)+4a\left(1+\frac{3a}{x}\right)-\frac{4a^2}{x}=0;$$

$$\therefore\; y-x+3a+\frac{11a^2}{x}=0.$$ Plate V. fig. 25.

124. In all these cases it must be observed that the first approximation gives, of all straight lines which can be drawn through the origin, that particular line which lies nearest to the infinite branch considered. The second approximation is represented by moving the line first found parallel to itself, until it becomes nearer to the curve than any other straight line which can be drawn.

The corresponding process with regard to points on a curve at a finite distance would be the following:

If P be a point on a curve, PT a tangent at P, PT lies nearer to the curve than any line which can be drawn through P, *i.e.*, if a point Q moves up to P, the perpendicular distance from Q on PT will be ultimately less than for any other line through P.

PLATE V.

A line OP drawn through the origin is nearer to Q than any line through O making a finite angle with OP; also the distance of Q from PT is of the second order of small quantities such as PQ, while that of Q from OP is generally of the first order, being of the second order when PT happens to pass through the origin.

If P move off to an infinite distance, we have at once the case of OP twisting round P until it coincides with PT, replaced by OP moving parallel to itself until it coincides with the asymptote; at first passing through only one point at infinity and afterwards through two consecutive points, the condition of an asymptote being thus satisfied.

125. The general statement of the process may be now made thus:

Let the rationalized equation of the curve be arranged in homogeneous functions of x and y in descending order of dimensions;

$$F_n(x, y) + F_{n-1}(x, y) + F_{n-2}(x, y) + \ldots = 0.$$

If x and y are both capable of indefinite increase, remaining of the same order of magnitude, along the infinite branches these functions are in descending order of magnitude.

The first approximation is $F_n(x, y) = 0$, which represents, generally, n straight lines through the origin which are parallel to the asymptotes.

Consider one solution of the equation, $ax \mp by$, and suppose first that the factor $ax - by$ only appears once, so that $F_n(x, y) \equiv (ax - by) f_{n-1}(x, y)$, where $f_{n-1}(x, y)$ is not zero, when ax is written for by.

The second approximation gives

$$ax - by + \frac{F_{n-1}(b, a)}{f_{n-1}(b, a)} = 0, \text{ Art. 3, (3), and Art. 50.,}$$

which is the asymptote parallel to $ax = by$, being the straight line which more nearly coincides with the curve at an infinite distance than any other straight line drawn in that direction.

PLATE V.

126. If there be no function of the $\overline{n-1}]^{\text{th}}$ degree, $ax = by$ is an asymptote.

In this case the next approximation is easily made, supposing $F_{n-r}(x, y)$ the first function which follows $F_n(x, y)$, for

$$ax - by + \frac{x^{r-1} F_{n-r}(x, y)}{f_{n-1}(x, y)} \frac{1}{x^{r-1}} = 0,$$

whence the next approximation gives

$$ax - by + \frac{b^{r-1} F_{n-r}(b, a)}{f_{n-1}(b, a)} \frac{1}{x^{r-1}} = 0,$$

which determines the side on which the curve lies.

127. Thus, in the hyperbola whose equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

the asymptotes $\frac{x}{a} \pm \frac{y}{b} = 0$ pass through the centre,

and $$\frac{x}{a} \pm \frac{y}{b} = \frac{1}{\frac{x}{a} \mp \frac{y}{b}} = \frac{a}{2x} \text{ nearly},$$

which gives the position of the curve relative to the asymptotes.

128. In the case in which $F_{n-1}(x, y)$ exists, the approximation for the direction of the curve with reference to the asymptote proceeds thus, the asymptote being

$$ax - by + c = 0,$$

whence $x : y :: b : a + \frac{c}{x}$, in the term following $ax - by$,

$$ax - by + \frac{F_{n-1}\left(b, a + \frac{c}{x}\right)}{f_{n-1}\left(b, a + \frac{c}{x}\right)} + \frac{bF_{n-2}(b, a)}{f_{n-1}(b, a)} \frac{1}{x} = 0 \ldots \text{(i)},$$

whence by neglecting terms in $\frac{1}{x^2}$,

$$ax - by + c + d\frac{c}{x} + \frac{e}{x} = 0.$$

PLATE V.

129. If $F_{n-2}(x, y)$ does not exist and the expansion of the third term in (i) has no term such as $\frac{dc}{x}$, the next approximation gives

$$ax - by + c + \frac{g}{x^2} + \frac{b^2 F_{n-3}(b, a)}{f_{n-1}(b, a)} \frac{1}{x^2} = 0;$$

such a case was that of the curve (2), Art. 123.

130. Consider now the case in which the factor $ax - by$ appears twice, so that

$$F_n(x, y) = (ax - by)^2 f_{n-2}(x, y).$$

If there be a function of the $(n-1)^{\text{th}}$ degree, the second approximation gives

$$(ax - by)^2 + \frac{F_{n-1}(b, a)}{bf_{n-2}(b, a)} . x = 0,$$

or, generally, for any arbitrary values of λ and μ,

$$(ax - by)^2 + \frac{F_{n-1}(b, a)}{(\lambda b + \mu a) f_{n-2}(b, a)} (\lambda x + \mu y) = 0.$$

In this case the curve has a parabolic branch in which $ax = by$ is the direction of the axis; or, if we write $bx + ay$ for $\lambda x + \mu y$, it is the axis itself of the parabola, the origin being the vertex.

This parabola is not generally a proper asymptote; for, by proceeding to the next approximation, it would appear that $by = ax + cx^{\frac{1}{2}} + d$,

since $$\frac{F_{n-1}(b, a + cx^{-\frac{1}{2}})}{bf_{n-2}(b, a + cx^{-\frac{1}{2}})} = c + dx^{-\frac{1}{2}}.$$

131. This argument will be seen by the curves following.

(1) $$x(y - x)^2 = ay^2.$$

Writing x for y in the second term, the asymptote is

$$(y - x)^2 = ax \text{ .... (i).}$$

For the next approximation

$$y - x = \sqrt{\frac{a}{x}} \{x + \sqrt{(ax)}\} = \sqrt{(ax)} + a \text{ ...... (ii).}$$

PLATE V.

For a further approximation

$$y - x = \sqrt{\frac{a}{x}}\,\{x + \surd(ax) + a\}$$

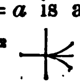
$$= \surd(ax) + a + a\sqrt{\frac{a}{x}}.$$

The difference between the curve and asymptote measured parallel to Oy is ultimately a for (i) and $a\sqrt{\frac{a}{x}}$ for (ii);

$$\therefore\ (y - x - a)^2 = ax$$

is the equation of a proper asymptotic parabola.

The two dotted lines represent the two parabolas, (i) and (ii) and shew that, for the general form of the curve, there is no necessity to obtain a proper asymptote.

$x = a$ is an asymptote, and the form near the origin is $x^3 = ay^2$  . Fig. 8.

(2) $$y\,(y - x)^2\,(y + 2x) = 9cx^3.$$

For the asymptote corresponding to $(y - x)^2$, writing x for y in the term following $(y - x)^2$,

$$(y - x)^2 = 3cx \text{ .... (i).}$$

The next approximation would give, writing $x + \surd(3cx)$ for y in the second term,

$$(y - x)^2 = \frac{9cx^3}{\{x + \surd(3cx)\}\,\{3x + \surd(3cx)\}},$$

$$= \frac{3cx}{1 + \frac{4}{3}\surd(3cx^{-1})},$$

$$\therefore\ y - x = \surd(3cx)\,\{1 - \tfrac{2}{3}\surd(3cx^{-1})\},$$

$$\therefore\ (y - x + 2c)^2 = 3cx \text{ .... (ii).}$$

The first parabola (i) is sufficient for the direction of the branches.

The second (ii) is a proper asymptote.

For an exercise in approximation, it may be shewn that a parabolic asymptote with a still closer contact is

$$(y - x + 2c)^2 = 3c\,(x + \tfrac{4}{2}c).$$

PLATE V.

The asymptote parallel to $y + 2x$ is

$$y + 2x = -\tfrac{1}{2}c\,;$$

$2y = 9c$ is another asymptote, which meets the curve $y^2 = 3x^2$.

The form near the origin is

Fig. 4.

$$y^4 = 9cx^3$$  .

132. If $F_{n-1}(x, y)$ have also a factor $ax - by$, the exis of this function does not cause a parabolic asymptote c ponding to the factor $(ax - by)^2$ of $F_n(x, y)$.

The equation may be written

$$(ax - by)^2 f_{n-2}(x, y) + (ax - by)\,\phi_{n-2}(x, y) + F_{n-2}(x, y) + .$$

so that the second approximation gives

$$(ax - by)^2 + \frac{\phi_{n-2}(b, a)}{f_{n-2}(b, a)}(ax - by) + \frac{F_{n-2}(b, a)}{f_{n-2}(b, a)} = 0,$$

which is the equation of two parallel rectilinear asymp real, coincident, or imaginary, $(ax - by + \alpha)(ax - by + \beta$

133. The process for finding the side on which the lies for the asymptote, when the equation is $ax - by +$ is to write the equation in the form

$$(ax - by)^2 + \frac{\phi_{n-2}(b, a + \alpha x^{-1})}{f_{n-2}(b, a + \alpha x^{-1})}(ax - by) + \frac{F_{n-2}(b, a + \alpha x}{f_{n-2}(b, a + \alpha x}$$

$$+ \frac{b\,.\,F_{n-3}(b, a)}{f_{n-2}(b, a)}\,\frac{1}{x} = 0,$$

in which it is easily seen that

$$\frac{\phi_{n-2}}{f_{n-2}} = \alpha + \beta + \text{terms in } \frac{1}{x},$$

$$\frac{F_{n-2}}{f_{n-2}} = \alpha\beta + \text{terms in } \frac{1}{x},$$

and $\quad (ax - by + \alpha)(ax - by + \beta) + (ax - by)\,\dfrac{c}{x} + \dfrac{d}{x} = 0\,;$

PLATE V.

therefore $$ax - by + \alpha + \frac{\alpha c}{\alpha - \beta}\,\frac{1}{x} + \frac{d}{x} = 0,$$

with variations if we have to proceed to higher powers of x.

134. If there be no function of the $(n-1)^{\text{th}}$ degree the second approximation gives

$$(ax - by)^2 + \frac{F_{n-2}(b, a)}{f_{n-2}(b, a)} = 0,$$

two parallel asymptotes, real or imaginary.

If the real asymptotes be $(ax - by)^2 = c^2$, the side of $ax - by = c$ on which the curve lies is found from

$$(ax - by)^2 + \frac{F_{n-2}(b, a - cx^{-1})}{f_{n-2}(b, a - cx^{-1})} = 0,$$

$$\text{or}\quad (ax - by)^2 - c^2 + \frac{d}{x^r} = 0,$$

$$ax - by - c + \frac{d}{2cx^r} = 0.$$

135. The following curves afford examples of these peculiarities:

(1) $$y(y - x)^2(y + 2x) = 3c^2x^2.$$

The second approximation to $y = x$ gives

$$(y - x)^2 = c^2,$$

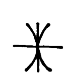
two parallel asymptotes, which meet the curve where

$$y^2 + 2xy = 3x^2,$$

hence $y = -3x$ gives the only real point of intersection with the curve.

The other asymptotes are $y = 0$, $y + 2x = 0$,

The curve is symmetrical in opposite quadrants; and near the origin

$$y^4 = 3c^2x^2$$  Fig. 5.

(2) $$(x + 2y)(x - y)^2 - 6a^2(x + y) = 0.$$

Here $(x - y)^2 = 4a^2$ gives two asymptotes, also $x + 2y = 0$ is an asymptote.

PLATE V.

The equation may therefore be written

$$(x+2y)(x-y-2a)(x-y+2a)-2a^2(x-y)=0,$$

$$x=0,\ y=0,\ \text{or}\ \pm\sqrt{3}\,a,$$

$$y=0,\ x=0,\ \text{or}\ \pm\sqrt{6}\,a,$$

$$\text{near}\ (0,0),\ x+y=0;$$

Fig. 6. second approximation $x+y=\dfrac{2y^3}{3a^2}$.

(3) $$(x+2y)^2(x-y)^2-a^3(x+y)=0.$$

The asymptotes are $(x-y)^2=0$, and $(x+2y)^2=0$.

For the position of the curve relative to the asymptotes

$$(x-y)^2=\frac{2a^3}{9x},$$

$$(x+2y)^2=\frac{2a^3}{9x};$$

Fig. 7. $x+y=0$ is the tangent at the origin.

(4) $$2x(x-y)^2-3a(x^2-y^2)+4a^2y=0.$$

For the asymptotes parallel to $x-y=0$,

$$(x-y)^2-3a(x-y)+2a^2=0;$$

$$\therefore\ x-y=a,\ \text{or}\ 2a.$$

For the asymptote parallel to Oy

$$2x+3a=0,$$

and the equation may therefore be written

$$(2x+3a)(x-y-a)(x-y-2a)+5a^2(x-y)-6a^3=0.$$

The cross asymptotes do not meet the curve except at an infinite distance.

Where $2x+3a=0$, $10y=-27a$,

...... $x=0$, $y=0$, or $-\frac{4}{3}a$,

...... $y=0$, $x=0$, or $\frac{3}{2}a$,

near $(0,0)$, $-3x^2+4ay=0$.

These considerations are sufficient to give the form of

Fig. 8. the curve.

PLATE V.

136. Although I consider this method of obtaining the cross asymptotes to be the best in almost every case, when its practical application is well understood, it is well to observe that in some cases a more direct method produces the asymptotes more readily.

Thus, when y can be expressed explicitly in terms of x, it can be expanded in descending powers of x, as in the following curve:

$$y = \frac{(x^2+x+1)(x-2)}{x.(x+1)},$$

$$y = x\left(1+\frac{1}{x}+\frac{1}{x^2}\right)\left(1-\frac{2}{x}\right)\left(1-\frac{1}{x}+\frac{1}{x^2}\ldots\right)$$

$$= x\left(1+\frac{1}{x}-\frac{2}{x}-\frac{1}{x}\right), \text{ for the asymptote,}$$

$$= x-2.$$

For the next approximation to the curve,

$$y = x\left\{1+\frac{1}{x(x+1)}\right\}\left(1-\frac{2}{x}\right)$$

$$= x\left(1-\frac{2}{x}+\frac{1}{x^2}\right)$$

$$= x-2+\frac{1}{x}.$$

The other process would be

$$x^2(y-x)+xy+x^2+x+2=0;$$

first approximation $y-x=0$,

second .... $y-x+2=0$,

third .... $y-x+\frac{y}{x}+1+\frac{1}{x}=0$,

$$\text{or } y-x+1-\frac{2}{x}+1+\frac{1}{x}=0;$$

$$\therefore\ y = x-2+\frac{1}{x}.$$

PLATE V. The other asymptotes are $x=0$ and -1;

$$\text{near } (0, \infty,) \quad x = -\frac{2}{y},$$

$$\ldots\ldots (-1, \infty), \quad x = \frac{3}{y},$$

Fig. 9. $$\ldots\ldots (2, 0), \quad y = \tfrac{7}{6}(x-2).$$

137. In tracing curves, it should be noticed by the student that, if an asymptote be looked upon as a tangent to a curve at a point infinitely distant, when this point is not a singular point of any kind, the curve lies on opposite sides at the two ends of the asymptote.

Thus, the axis of x is an asymptote of the common hyperbola, whose equation is $xy = c^2$, meeting the curve at two points at an infinite distance; and the side of the asymptote on which the curve lies is determined by $y = \frac{c^2}{x}$, shewing that, since y is of the same sign as x, the curve is on opposite sides of the asymptote.

If there be three points at an infinite distance, when the asymptote is a tangent at a point of inflexion at an infinite distance the curve lies on the same side at both ends; thus, if $(x-a)y^2 = c^3$ be the equation of the curve, $x-a=0$ is an asymptote meeting the curve at three points at an infinite distance, and since $x - a = \frac{c^3}{y^2}$, the curve lies on the same side at both ends. Fig. 10.

But, when the asymptote is a tangent to a branch of a double point at an infinite distance, two of the three points belong to the branch which is touched by the asymptote, and one to the other branch, and in this case the curve lies on opposite sides of each of the two asymptotes.

Thus $(x-a)(x-b)y = c^3$ is a curve which has two asymptotes $x=a$ and b.

$x=a$ meets the curve in three points at an infinite distance, one of these points is the point in which it meets the branch which touches $x=b$, and the other two are the con-

secutive points, by passing through which, it is itself a tangent to another branch. PLATE V.

$x - a = \frac{c^3}{(a-b)\,y}$, being the next approximation, shews that the curve lies on opposite sides of the asymptote at the two ends. Fig. 11.

These considerations are useful, because, prior to proof, we ought to draw the curve in such a way that it shall leave the asymptote on opposite sides, and we are thus led to examine the points at a finite distance at which the curve crosses the asymptotes.

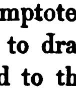
138. Take, as an example, the curve

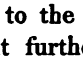
$$x^5 - ax^3y - bx^2y^2 + y^5 = 0.$$

Near (∞, ∞), the first approximation is $x + y = 0$, the second gives $x + y = \frac{1}{5}(b - a)$.

Near (0, 0), x, y of the same order of magnitude gives

$$ax + by = -\frac{b^5 - a^5}{b^4 a}\,x^2 \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

$x : y$ small gives $y^3 = bx^2$  .... (ii),

$y : x$ .... $x^2 = ay$  .... (iii).

Now, without further examination, we might be disposed to connect the forms already obtained, so that the curve would lie on the same side of the asymptote. But the consideration given above would lead us to draw the curve so as to cross the asymptote and proceed to the opposite side. If we tried to draw the curve without further calculation we should have to judge from its direction which of the branches would be more likely to cross. Thus if $b < a$ the branch (iii) would be more likely to cross than (i), since it has so close a contact with Ox. If $b > a$, the same reason would make us select the branch (ii).

We might then test by proceeding to the next approximation to the asymptote, or by absolutely finding the points

PLATE V.

Near (0, 0), $y^3 = 8ax^2$ ,

...... (∞, ∞), $y - x = \lim. \dfrac{8ax^3}{(y-4x)(y+2x)} = -\frac{8}{9}a,$

$$y - 4x = \tfrac{4}{9}a,$$

$$y + 2x = \tfrac{4}{9}a,$$

...... $(a, 0)$, $3y = 4(x - a)$.

To find where the asymptotes meet the curve:

(i) $$y - x = -\tfrac{8}{9}a,$$

$$(y-4x)(y+2x) = -9x^2;$$

$$\therefore\ y^2 - 2xy + x^2 = 0.$$

(ii) $$y - 4x = \tfrac{4}{9}a,$$

$$(y-x)(y+2x) = 18x^2;$$

$$\therefore\ y^2 + xy - 20x^2 = 0;$$

$$\therefore\ y + 5x = 0.$$

(iii) $$y + 2x = \tfrac{4}{9}a,$$

$$(y-x)(y-4x) = 18x^2;$$

$$\therefore\ y^2 - 5xy - 14x^2 = 0;$$

$$\therefore\ y - 7x = 0.$$

The first asymptote meets the curve at three infinitely distant points.

Fig. 17. (4) $$(y-x)^2(y+x)(y+2x) = 16a^4.$$

One of the asymptotes is double;

$$x = 0,\ \ y = \pm 2a,$$

$$y = 0,\ \ x = \pm 8^{\frac{1}{4}}a.$$

Fig. 18. The curve is symmetrical in opposite quadrants.

(5) $$(y-x)^2(y+x)(y+2x) = 6ax^3.$$

With this variation there is a parabolic asymptote,

$$(y-x)^2 = ax.$$

PLATE V.

The other asymptotes are

$$y + x = \tfrac{3}{2}a,$$

$$y + 2x = -\tfrac{2}{3}a.$$

Near the origin, $y^4 = 6ax^3$  ,

near $(3a, 0)$, $y = 2(x - 3a)$.

$y + x = \frac{3}{2}a$ meets the curve where

$$(y - x)^2 (y + 2x) = 4x^3,$$

$$\text{or } (y^2 - 4x^2)\, y + x^2 y - 2x^3 = 0;$$

$$\therefore\ y^2 - xy - 2x^2 = 0,$$

or $y - 2x = 0$ gives the only finite point.

$y + 2x = -\frac{2}{3}a$ meets the curve where

$$(y - x)^2 (y + x) = -9x^3,$$

$$\text{or } (y^2 - x^2)\, y - xy^2 + 10x^3 = 0,$$

$$y^2 (y + 2x) - 3xy\,(y + 2x) + 5x^2 (y + 2x) = 0;$$

$$\therefore\ y^2 - 3xy + 5x^2 = 0,$$

which gives two impossible points. Fig. 19.

140. I shall conclude this chapter by giving materials for tracing curves which have infinite branches of the various kinds which have been discussed, by which the student may practice himself in completing curves from the known forms of particular parts.

(1) $$x^5 - 2a^3xy + y^5 = 0.$$

Observe the symmetry with respect to $x = y$.

Near $(0, 0)$, $x^4 = 2a^3y$ and $y^4 = 2a^3x$,

...... (∞, ∞), $x + y = 0$;

$$x = y = a.$$ Fig. 20.

(2) $$x^5 - 5ax^2y^2 + y^5 = 0.$$

Near $(0, 0)$, $x^3 = 5ay^2$ and $y^3 = 5ax^2$,

...... (∞, ∞), $x + y = \lim. \dfrac{5ax^2y^2}{x^4 - x^3y + \ldots + y^4} = a$;

$$x = y = \tfrac{5}{2}a.$$ Fig. 21.

PLATE V.

(3) $$x^5 - a^2x^2y - b^2xy^2 + y^5 = 0.$$

The curve is symmetrical in opposite quadrants.

Near (0, 0), $x : y$ small gives $y^5 = b^2x$,

$y : x$ small gives $x^3 = a^2y$,

x and y of the same order gives $a^2x + b^2y = 0$.

Near (∞, ∞), $y + x = 0$.

Fig. 22. The figure is drawn for $a > b$.

(4) $$y^2(3x - 4y)^3 - a^4x = 0.$$

This is symmetrical in opposite quadrants.

Near (0, 0), $64y^5 + a^4x = 0$,

...... $(\infty, 0)$, $y^2 = \dfrac{a^4}{27x^3}$,

Fig. 23. ...... (∞, ∞), $(3x - 4y)^3 = \dfrac{4a^4}{3y}$.

(5) $$\{x(y - x) - a^2\}^2 y^3 = a^7.$$

Observe that y cannot be negative.

Near $(0, a)$, $y = a + \frac{2}{3}x$,

...... (a, a), $\eta + 2\xi = 0$,

...... $(0, \infty)$, $x = \dfrac{a^2}{y} \pm \sqrt{\left(\dfrac{a^7}{y^5}\right)}$,

...... $(\infty, 0)$, $y = \left(\dfrac{a^7}{x^4}\right)^{\frac{1}{3}}$,

Fig. 24. ...... (∞, ∞), $y - x = \dfrac{a^2}{x} \pm \sqrt{\left(\dfrac{a^7}{x^5}\right)}$.

(6) $$4a^2(y - x)^2(y + x) - 8ax^3(y - x) + x^5 + y^5 = 0.$$

Near (∞, ∞), $x + y + \frac{16}{5}a = 0$;

near the origin, where x and y are of the same order,

$$y - x = 0, \text{ and } y + x = 0.$$

For the next approximation to $y = x$,

$$(y - x)^2 - \frac{x^2}{a}(y - x) + \frac{x^4}{4a^2} = 0,$$

$$\text{or } \left(y - x - \frac{x^2}{2a}\right)^2 = 0;$$

for a third approximation PLATE V.

$$(y-x)^2-\frac{x^2}{a}\left(1-\frac{x}{4a}\right)(y-x)+\frac{x^4}{4a^2}\left(1+\frac{5x}{4a}\right)\left(1-\frac{x}{4a}\right)=0\,;$$

$$\therefore\ \left(y-x-\frac{x^2}{2a}\right)^2+\frac{x^5}{8a^3}+\frac{x^5}{4a^3}=0\,;$$

$$\therefore\ y=x+\frac{x^2}{2a}+\left(-\frac{3x^5}{8a^3}\right)^{\frac{1}{2}}.$$

For the other branch through the origin,

$$16a^2x^2(y+x)+16ax^4=0\,;$$

$$\therefore\ y+x=-\frac{x^2}{a}.$$

$$\text{If } y=0,\ \ 4a^2x^3+8ax^4+x^5=0\,;$$

$$\therefore\ x^2+8ax+16a^2=12a^2\,;$$

$$\therefore\ x=-2a\,(2\pm\sqrt{3})=-2a\tan\tfrac{5}{12}\pi\ \text{ or }\ -2a\tan\tfrac{1}{12}\pi.$$

To find the side of the asymptote on which the curve lies, let

$$x+y+\tfrac{16}{5}a=\alpha\,;$$

$$\therefore\ x^5-(x+\tfrac{16}{5}a-\alpha)^5+8ax^3(2x+\tfrac{16}{5}a-\alpha)=0,$$

$$\text{or }\ 5\alpha x^4-10\,(\tfrac{16}{5}a)^2x^3+8\,.\,\tfrac{16}{5}a^2x^3=0,$$

$$\alpha x=24\,.\,\tfrac{16}{5}a^2.$$

Fig. 25.

(7) $$(x^2-y^2)^2-4y^2+y=0.$$

Expanding in descending powers of y,

$$x^2=y^2\pm2y\left\{1-\frac{1}{8y}-\ldots\right\},$$

$$\pm x=y\left\{1\pm\frac{2}{y}\mp\frac{1}{4y^2}\ldots\right\}^{\frac{1}{2}},$$

$$=y\left\{1\pm\frac{1}{y}\mp\frac{1}{8y^2}-\frac{1}{8}\,\frac{4}{y^2}\ldots\right\};$$

$$\therefore\ x=y\pm1-(\tfrac{1}{2}\pm\tfrac{1}{8})\,\frac{1}{y}\ldots,$$

$$\text{and }\ -x=y\pm1-(\tfrac{1}{2}\pm\tfrac{1}{8})\,\frac{1}{y}\ldots,$$

PLATE V.

which give the four asymptotes, and the side on which the curve lies.

$$\text{Near the origin, } x^4 + y = 0,$$

$$\text{near } (\tfrac{1}{4}, \tfrac{1}{4}),\ \{\tfrac{1}{2}(\xi - \eta)\}^2 - \eta = 0, \text{ or } \xi^2 = 4\eta.$$

The curve is symmetrical with respect to Oy, which it cuts at the origin, and where

$$y^3 - 4y + 1 = 0.$$

The solutions of this equation are two positive and one negative, and the equation of the curve may be written

$$x^2(2y^2 - x^2) = y(y-\alpha)(y-\beta)(y+\gamma),\quad \alpha > \beta.$$

$$\text{Near } (0, \alpha),\quad 2\alpha^2 x^2 = \alpha(\alpha-\beta)(\alpha+\gamma)\eta,$$

$$\ldots\ldots (0, \beta),\quad 2\beta^2 x^2 = -\beta(\alpha-\beta)(\beta+\gamma)\eta,$$

$$\ldots\ldots (0, -\gamma),\quad 2\gamma^2 x^2 = -\gamma(\alpha+\gamma)(\beta+\gamma)\eta.$$

Fig. 26. The asymptotes meet the curve where $y = \frac{1}{3}$, and $-\frac{1}{5}$.

(8) $y(y-x)^2(y-2x) + 3a(y-x)x^2 - 2a^2x^2 = 0.$

There is an asymptote $2y + 3a = 0$;

near (∞, ∞), $y - x = a$ or $2a$, and $y - 2x = -\frac{3}{2}a$.

The equation may be written

$$(y-x)^4 - x^2(y-x-a)(y-x-2a) = 0,$$

which shews that the parallel asymptotes do not intersect the curve except at infinity, and that no part of the curve lies between them.

$$\text{Near the origin, } y^4 = 2a^2x^2,$$

$$\text{near } (-\tfrac{3}{2}a, 0),\ y(-2x+3a) - 3a(x+\tfrac{3}{2}a) = 0;$$

Fig. 27.

$$\therefore\ 13y = 9\xi.$$

(9) $$y^2x(y-x) - ay^3 - byx^2 + c^2x^2 = 0,$$

$$\text{or } (x-a)y^3 - x^2y^2 - bx^2y + c^2x^2 = 0,$$

$$\text{or } (y^2 + by - c^2)x^2 - y^3x + ay^3 = 0.$$

The asymptotes parallel to the axes are $x = a$,

$$\text{second approximation, } x = a + \frac{a^2}{y},$$

PLATE V.

and $y^2+by-c^2=0$, or $(y-\alpha)(y+\beta)=0$,

$$\text{second approximation, } y = \alpha + \frac{\alpha^3}{\alpha+\beta}\cdot\frac{1}{x},$$

$$\text{and } y = -\beta + \frac{\beta^3}{\alpha+\beta}\cdot\frac{1}{x}.$$

The cross rectilinear asymptote is

$$y-x=a+b,$$

which meets the curve where

$$(a+b)\,y^2x - ay^3 - byx^2 + c^2x^2 = 0,$$

$$\text{or } a\,(a+b)\,y^2 - b\,(a+b)\,xy - c^2x^2 = 0,$$

the roots of which are of opposite signs;

$$\text{near the origin, } c^2x^2 = ay^3.$$

The curve passes through the intersection of the asymptotes parallel to the axes; it is drawn for the case in which $c^2 > a^2 - b^2$. Fig. 28.

Note. If $a > b$, the curve cuts the cross asymptote in only one point when $c^2 = a^2 - b^2$; and, if $c^2 < a^2 - b^2$, the curve cuts it below the axis of x.

(10) $$x^7 - x^3y^4 + a^4y^3 - ax^2y^4 = 0.$$

$x=0$ is an asymptote, and the next approximation gives $x^2 = \dfrac{a^3}{y}$.

$x+a=0$ is also an asymptote, meeting the curve where $y=a$; the next approximation is $x+a=\dfrac{a^2}{y}$.

The other asymptotes can be found, as well as the side on which the curve lies, from

$$x^5-(x+a)\,y^4=0;$$

$$\therefore\ y = \pm x\left(1+\frac{a}{x}\right)^{-\frac{1}{4}},$$

$$= \pm x\left(1-\frac{1}{4}\cdot\frac{a}{x}+\frac{5}{32}\,\frac{a^2}{x^2}\right),$$

$$= \pm\left(x-\frac{1}{4}\,a+\frac{5}{32}\,\frac{a^2}{x}\right).$$

PLATE V.

Near the origin, $x^7 + a^4y^3 = 0$,

near $(-a, a)$, $2\xi + \eta = 0$,

Fig. 29. ...... (a, a), $2\xi - 5\eta = 0$.

(11) $x^2y - y^2x = a\,\overline{x-b}|^2 - b\,\overline{y-a}|^2$.

The equation may be written

$$(x-b)\,y^2 + (2ab - x^2)\,y + a\,(\overline{x-b}|^2 - ab) = 0,$$

or $(y-a)\,x^2 + (2ab - y^2)\,x + b\,(\overline{y-a}|^2 - ab) = 0.$

$$x = 0,\ \ y = a \pm \sqrt{(ab)},$$

$$y = 0,\ \ x = b \pm \sqrt{(ab)},$$

$$x = b,\ \ y = \infty,\ \text{or}\ \frac{a^2}{2a-b},$$

$$y = a,\ \ x = \infty,\ \text{or}\ \frac{b^2}{2b-a},$$

$$x = y,\ x^2 = ab.$$

Near $x = 0$, $y = a \pm \sqrt{(ab)} \pm \dfrac{b - a \pm 2\sqrt{(ab)}}{2b}\sqrt{\dfrac{a}{b}}\,x$,

...... $\{\sqrt{(ab)}, \sqrt{(ab)}\}$, $\{3\sqrt{(ab)} - 2a\}\,\xi = \{3\sqrt{(ab)} - 2b\}\,\eta$,

...... (∞, ∞), $x - y = a - b$,

Fig. 30. which may be shewn to meet the curve where $\dfrac{x}{b} + \dfrac{y}{a} = 3$.

VII.

(1) $x(y-x)^2 = c^3$.

Find the tangents to the curve which pass through $(c, 0)$, shew that the point of contact of one is $(\frac{1}{4}c, \frac{9}{4}c)$, and that the other is inclined to Ox at an angle $\tan^{-1}\frac{3}{2}$.

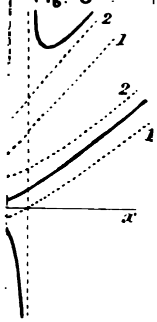
(2) $x^5 - a^3x^2 + y^5 = 0$.

(3) $x^5 - a^2x^3 + y^5 = 0$.

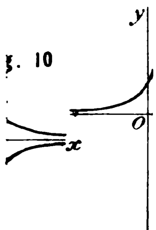
(4) $x^5 - ax^4 + y^5 = 0$.

Show that the point at which the curve is parallel to Ox has coordinates $\frac{4}{5}a$ and $\frac{3}{5}a$ nearly.

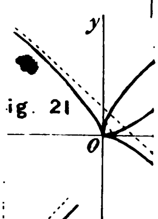
Fig. 3

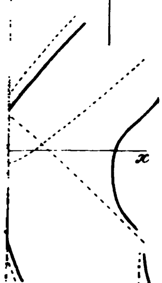


g. 10



ig. 21





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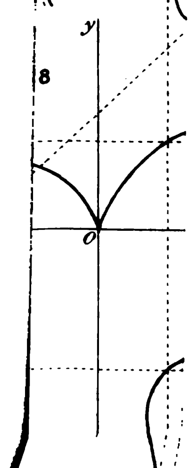


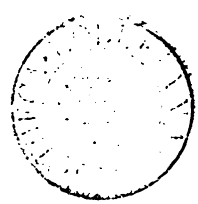


PLATE V.

(5) $y^4 + 2axy^2 - x^4 = 0$.

Show that the asymptotes are intersected by the curve where $y = (\pm \sqrt{2} - 1)\,x$.

(6) $x^7 - ax^5y - y^7 = 0$.

(7) $x^7 - ax^4y^2 - y^7 = 0$.

(8) $xy\,(y-x)\,(y-2x) = c^4$.

(9) $y\,(y-x)^2\,(y+2x) = c^4$.

(10) $y\,(y-x)^2\,(y+2x) = c^3x$.

Prove that the lines joining the points at which the curve is parallel to Oy are inclined to Ox at angles whose tangents are $\sqrt{2}\,\sin\dfrac{\pi}{12}$ and $-\sqrt{2}\,\cos\dfrac{\pi}{12}$.

(11) $(x-y)^2\,(x+y)\,(x^2+y^2) = a^4x$.

(12) $(x-y)^2\,(x+y)\,(x^2+y^2) = a^3x^2$.

(13) $(x-y)^2\,(x+y)\,(2x+y) - a^3y = 0$.

(14) $(x-y)^2\,(x+y)\,(2x+y) - a^2y^2 = 0$.

(15) $(y^2-x^2)^2 - a^2y^2 + b^2x^2 = 0$.

Discuss the cases of $a >$ and $< b$, and in the latter examine $b >$ and $< a\sqrt{2}$.

If a straight line be drawn parallel to one of the asymptotes meeting the curve in points, such that the lines drawn to them from the origin are at right angles, shew that they are equally inclined to one of the tangents at the origin.

(16) $x^4 - y^4 - 2ax^2y + a^2y^2 + a^4 = 0$.

Shew that the curvatures at the points of intersection with Oy, and at (a, a) are in the ratio $1 : \sqrt{5}$.

(17) $x^4 - y^4 - 3a^2xy + ax^3 + ay^3 = 0$.

(18) $yx\,(y^2-x^2) - ay^3 - byx^2 + c^2x^2 = 0$.

CHAPTER VIII.

CURVILINEAR ASYMPTOTES.

PLATE VI.

141. The only cases of infinite branches which remain to be discussed are those in which, when x and y are both infinite, they are not of the same order of magnitude, so that $x : y$ or $y : x$ vanishes ultimately.

The equation of a curve being given, it may, generally, if it contain many terms, be simplified considerably by considerations similar to those given in Art. 95 for the investigation of branches through the origin.

Suppose that we intend to examine whether $x : y$ can ultimately vanish in any infinite branch of the curve.

(1) All homogeneous functions of x and y may be replaced by the term which involves the highest power of y.

(2) The coefficient of any power of x being a function of y, the term involving the highest power of y is the only term which need be retained.

(3) A similar observation holds with respect to the coefficient of any power of y.

(4) If two terms $ax^s y^{r+1}$, $bx^t y^r$ remain, since their ratio is $ay : bx^{t-s}$, $t - s$ must be > 1, therefore the indices of x must ascend by steps of 2 or more, if the equation be rational.

These considerations reduce the equation to a comparatively small number of terms, and it must then be tried whether, on neglecting one or more terms, a relation such as $y = x^r$, $r > 1$, is obtained, and whether with this relation the terms rejected are smaller than those retained.

142. In tracing curves, the case of x being great with respect to y, or *vice versâ*, in any infinite branch is always

to be examined carefully, for with a cursory glance, which is generally sufficient, such a branch often escapes notice, and is only eventually detected by the impossibility of uniting the parts which have been discovered.

143. The examples which I have given in this chapter should be followed out as far as the asymptotes are concerned, which I have for this reason placed at the commencement in each case; the complete curve can be referred to for the position of the infinite branches, and the discussion of the form at a finite distance from the origin can be deferred for the present.

(1) $$x^6 + 2a^2x^3y - b^3y^3 = 0.$$

There are no asymptotes parallel to the axes; the only infinite branch is where $y : x$ is large, the form being $x^2 = by$, where $x^3y \propto x^5$, and may be neglected compared with the term x^6 retained in the first approximation.

The parabola, $x^2 = by$, is not a proper asymptote, but is sufficient to guide us as to the direction of the infinite branches.

The proper asymptotes may be obtained as an exercise in approximation. Thus

$$by = x^2\left(1 + \frac{2a^2y}{x^3}\right)^{\frac{1}{3}},$$

$$= x^2\left(1 + \frac{2a^2y}{3x^3} - \frac{4a^4y^2}{9x^6}\right);$$

second approximation $= x^2\left(1 + \dfrac{2a^2}{3bx}\right);$

third .... $= x^2\left\{1 + \dfrac{2a^2}{3bx}\left(1 + \dfrac{2a^2}{3bx}\right) - \dfrac{4a^4}{9b^2x^2}\right\},$

$$= x^2 + \frac{2a^2}{3b}x.$$

The proper parabolic asymptote is therefore

$$\left(x + \frac{a^2}{3b}\right)^2 = b\left(y + \frac{a^4}{9b^3}\right);$$

PLATE VI.

Note. The curve is parallel to Oy at the intersection of the curves

$$y(x-a)^4 = \tfrac{3}{2}a^5 \text{ and } xy^3 = \tfrac{1}{2}a^4,$$

and parallel to Ox at the intersection of

$$4(x-a)^3 = ay^2 \text{ and } (3x+a)y^3 + 4a^4 = 0.$$

(5) $$axy^3 + (x-a)^3 y^2 + a^5 = 0.$$

Near $(0, \infty)$, $x = \dfrac{a^2}{y}$,

...... $(\infty, 0)$, $y^2 = -\dfrac{a^5}{x^3}$,

...... (∞, ∞), $ay + x^2 = 0$,

...... $(0, a)$, $\eta = 2x$,

...... $(0, -a)$, $\eta = -x$,

...... $(a, -a)$, $3\eta = \xi$.

The curve is parallel to Ox at the intersection of

$$ay + 3(x-a)^2 = 0,$$

$$\text{and } (2x+a)y^3 + 3a^4 = 0,$$

which gives three negative values of y, and two positive, one negative value of x.

Fig. 5.

(6) $$a^3(y+x) - 2a^2x(y+x) + x^4 = 0;$$

or it may written

$$a^2(a-2x)y + x(x-a)(x^2+ax-a^2) = 0.$$

Near $(\tfrac{1}{2}a, \infty)$, $x - \tfrac{1}{2}a = \dfrac{a^2}{32y}$,

...... (∞, ∞), $-2a^2y + x^3 = 0$,

...... $(0, 0)$, $y + x + \dfrac{x^4}{a^3} = 0$;

where $y=0$, $x=a$, and $\tfrac{1}{2}(\pm\sqrt{5}-1)a$, $=\alpha a$ and βa suppose.

Near $(a, 0)$, $y = \xi$,

...... $(\alpha a, 0)$, $(1-2\alpha)y + \alpha(\alpha-1)(\alpha-\beta)\xi = 0$,

PLATE VI.

$$\text{and } x^2 - x = 1 - 2x;$$

$$\therefore\ y + \sqrt{5}\,\xi = 0,$$

$$\text{near } (\beta a, 0),\ \ y - \sqrt{5}\,\xi = 0.$$ Fig. 6.

(7) $$y^4 - 2(3x - 4a)\,ay^2 + a^2x^2 = 0.$$

The equation may be written

$$(y^2 - 3ax + 4a^2)^2 = 8a^2(x - 2a)(x - a),$$

near (∞, ∞), $y^2 = (3 \pm 2\sqrt{2})\,ax$, for a first approximation; the proper asymptotes are

$$y^2 = (3 \pm 2\sqrt{2})\,a\,(x \mp a\sqrt{2}).$$

The origin is a conjugate point, and x cannot give real values of y when it is between a and $2a$.

$x = 2a$ is a tangent where $y = \pm a\sqrt{2}$. Fig. 7.

(8) $$x^4 - 3ax^2y + 2a^2y^2 - ay^3 = 0.$$

$$\text{Near } (\infty, \infty),\ \ x^4 - ay^3 = 0;$$

near the origin, ay^3 may be rejected, therefore $x^2 = ay$, or $2ay$;

$$\text{near } (0, 2a),\ \ x^2 = -\tfrac{4}{3}a\eta;$$

$$\ldots\ldots\ (\sqrt{6a}, 2a),\ \ 11\eta = 6\sqrt{6}\,\xi,$$

observe that y cannot be negative. Fig. 8.

(9) $$x^7 - x^4y^3 + a^2y^4 - axy^5 = 0.$$

$$\text{Near } (0, \infty),\ \ x = \frac{a^2}{y};$$

$\ldots\ldots$ (∞, ∞), x and y of the same order,

$$x - y - \tfrac{1}{3}a = 0;$$

$x : y$ small, x^7 and y^4 may be rejected;

$$\therefore\ x^3 + ay^2 = 0.$$

Near the origin, $y : x$ small gives $x^7 + a^2y^4 = 0$. Fig. 9.

(10) $$a(x^5 + y^5) - a^2x^2y + x^2y^4 = 0.$$

Near (∞, ∞), $x : y$ large gives $ax^3 + y^4 = 0$; $y : x$ large gives $ay + x^2 = 0$.

PLATE VI.

Near the origin, if $x : y$ be small,

$$ay^4 - a^2x^3 + x^2y^3 = 0;$$

therefore $y^4 - ax^3 = 0$, making x^2y^3 of order $x^{4+\frac{1}{4}}$.

If $y : x$ be small, y^5 and x^2y^4 may be rejected, giving

Fig. 10. $x^2 = ay$. Where $x = y$, $x^2 + 2ax = a^2$.

(11) $$x^7 - a^3x^3y(x-y) + a^5(x-y)^2 = 0.$$

Near (∞, ∞), $x^5 + a^3y^2 = 0$;

and near the origin, since we have

$$y - x + \frac{x^3y}{2a^3} = \pm\frac{x^3y}{2a^3}\left(1 - \frac{4x^3}{ay^3}\right)^{\frac{1}{2}},$$

$$= \pm\frac{x^3y}{2a^3}\left(1 - \frac{2x^3}{ay^3} - \ldots\right),$$

$$y - x = -\frac{x^4}{a^3}, \text{ or } -\frac{x^2}{2a^2};$$

near $(-a, 0)$, $a^2.5a^4\xi - a^6y + 2a^6y = 0$,

Fig. 11. or $3\eta + 5\xi = 0$.

(12) $$ay^2(y-a) - x^2(y^2 - a^2) + 2axy^2 - x^4 = 0.$$

Near (∞, ∞), y large compared with x gives

$$ay^3 - x^2y^2 = 0, \text{ or } ay = x^2;$$

near the origin, $y^2 = x^2$;

near $(0, a)$, $a^3\eta + 2a^3x = 0$.

The next approximation gives

$$a^3(\eta + 2x) = -4ax^3.$$

The equation may be written in the form

$$ay^3 - (x-a)^2y^2 - x^2(x^2 - a^2) = 0,$$

near $(a, 0)$, $ay^3 - a^2 2a\xi = 0$,

...... $(-a, 0)$, $-4a^2y^2 + 2a\xi = 0$,

The tangent at $(0, a)$, meets the curve again at the

Fig. 12. point $(\frac{4}{3}a, -\frac{5}{3}a)$ only.

Note. The proper asymptote is $a(y-a) = (x-a)^2$, which meets the curve in only one point at a finite distance.

PLATE VI.

(13) $$c^6y^2 - (a+b)\,c^3x^3y + abx^6 - c^3x^3y^2 = 0.$$

The equation may be written

$$(c^3y - ax^3)(c^3y - bx^3) - c^3x^3y^2 = 0,$$

$$\text{or } c^3(c^3 - x^3)\,y^2 - \{(a+b)\,c^3y - abx^3\}\,x^3 = 0.$$

Near (c, ∞), $x = c\left(1 - \dfrac{a+b}{3y}\right)$,

...... (∞, ∞), $abx^3 - c^3y^2 = 0$,

...... $(0, 0)$, $c^3y = ax^3$, or bx^3. Fig. 13.

Note. The two asymptotes are cut by the curve at points equidistant from Ox.

(14) $$yx^2(y - x) - ay^3 - byx^2 + c^2x^2 = 0.$$

The equation may be written

$$yx^3 - (y^2 - by + c^2)\,x^2 + ay^3 = 0.$$

Near $(\infty, 0)$, $y = \dfrac{c^2}{x}$;

...... (∞, ∞), x and y of the same order,

$$y - x - a - b = 0,$$

this asymptote meets the curve at a finite distance, where

$$(a+b)\,yx^2 - ay^3 - byx^2 + c^2x^2 = 0,$$

$$(a+b)\,ay\,(y+x) - c^2x^2 = 0,$$

one of whose roots is positive, the other negative.

When y and x are of different orders

$$x^3 - yx^2 + ay^2 = 0;$$

$\therefore\ x^2 = ay$ is a parabolic asymptote.

Near the origin, $ay^3 = c^2x^2$  .

The figure includes the cases $c^2 = 2a(a+b)$, to which the branches marked b and d belong, since the curve meets the asymptote in one point only at a finite distance; Fig. 14.

also $c^2 > 2a(a+b)$, a and d being the branches,

and $c^2 < 2a(a+b)$, b and c being the branches.

PLATE VI.

(15) $$(y-x-a)\,y^3 - byx^2 + mabx^2 = 0.$$

$$\text{Near } (\infty, ma),\ y = ma - \frac{m^3a^3}{bx},$$

$$\ldots\ldots\ (\infty, \infty),\ y = x + a + b - \frac{(m+2)\,ab}{x}.$$

$$\text{If } x : y \text{ be large, } y^2 + bx = 0,$$

$$y : x \text{ cannot be large.}$$

$$\text{Near } (0, 0),\ mbx^2 = y^3,$$

$$\ldots\ldots\ (0, a),\ y - x - a = -(m-1)\,b\,\frac{x^2}{a^2}.$$

The asymptote $y = ma$ meets the curve where $x = (m-1)\,a$, the same point at which the tangent at $(0, a)$ meets it.

The cross asymptote meets the curve where

$$y^3 - yx^2 + max^2 = 0,$$

$$\text{or } (a+b)\,(y+x)\,y + max^2 = 0,$$

the roots of which are real and both negative, if $4ma < a + b$, in which case the curve cuts the asymptote in the compartment $x'Oy$ below the line $x + y = 0$.

The curve is parallel to Ox at the points where

$$x = 2\,(y-a).$$

To illustrate the different forms which the curve can assume, it is drawn for three cases

Fig. 15. $$b = a, \quad m > \tfrac{1}{2},$$

Fig. 16. $$b = a, \quad m > 1,$$

Fig. 17. $$\text{and } b = a\sqrt{3}, \quad m = 2;$$

the last being an intermediate case in which there is a multiple point, viz. where $x = (\sqrt{3}-1)^2\,a$, $y = (\sqrt{3}-1)\,b$.

Note. If $m < 1$, the curve bends upwards; if $m = 1$, there is a point of inflexion at $(0, a)$.

In figure 16, the points a, b, c are points in $2y - x - 2a = 0$, where the curve is parallel to Ox.

VIII.

(1) $y^2(x+1) = x^2 + 1$.

The curve is parallel to Ox, where $x = \sqrt{2} - 1$.

(2) $c^3y^2 = (x-a)^2(x-b)^3$.

(3) $x^2y + ay^2 - 2axy + ax^2 = 0$.

Shew that the curve cuts the rectilinear axis at an angle $\tan^{-1}\frac{8}{3}$.

(4) $x^4 - 3a^2xy + ay^3 + ax^3 = 0$.

(5) $y^3 - y^2 - 64x^6 - 36x^5 - 2x^4 = 0$.

(6) $x^3(y-x)^4 - a^2y^5 = 0$.

(7) $a(x^5 + y^5) - x^2y^4 = 0$.

(8) $a(x^5 + y^5) - xy^5 = 0$.

Prove that, if the tangent be drawn from $(a, 0)$, to the curve, the point of contact is at a distance $5a$ from the axis of y.

(9) $axy^3 - (x-a)^3y^2 + a^5 = 0$.

CHAPTER IX.

THE ANALYTICAL TRIANGLE. PROPERTIES OF THE ANALYTICAL TRIANGLE.

PLATE VI.

144. I HAVE now shewn how, in all cases of curves represented by equations containing a *moderate* number of terms, simpler curves can be found, which very nearly coincide with the curves near particular points, when they are at a finite distance, and nearly enough for practical purposes when they are at an infinite distance.

In this chapter I shall give some account of the *Analytical Triangle*, and its use as a machine for saving the trouble of the comparison of the magnitude of the different terms of the equation of a curve, at an infinite distance, and in the neighbourhood of the origin, when the curve passes through it.

145. The triangle is a modification of Newton's parallelogram, which was an arrangement of squares, like those on a chess-board, each square being appropriated to one of the terms of the general equation of any degree, as in the
Fig. 18. figure.

146. It is easily seen, by observing the squares which contain the terms of four and lower dimensions, that all the terms of a complete equation of any degree are contained in squares which occupy half of Newton's parallelogram; this circumstance led De Gua to replace the parallelogram by a triangle containing one more square on each side than the degree of the equation considered, which is represented
Fig. 19. in the figure for an equation of the fourth degree.

147. An equation is said to be placed upon the triangle, by making a cross, or some definite mark, in the centre of each square which corresponds to a term of the equation.

Thus, the equation

$$ay^4 + bx^2y^3 + cx^3y^2 + dx^4y + ex^2 + fxy + gx^2y^2 = 0$$

is placed on the triangle as in the figure. Fig. 20.

148. The property which makes the triangle so valuable as an analyzer, is that, if crosses be joined, so as to form a polygon $\alpha\beta\gamma\delta\epsilon$, exterior to which no cross lies, when the terms of the equation, which correspond to any side, are the only terms retained, the locus of the equation so formed is one or more simple parabolic curves, or straight lines, each of which is a first approximation to the form of the curve, *either* at an infinite distance, if all the rejected crosses lie on the same side of the line as the right angle, *or*, near the origin, if they lie on the opposite side, when the equation has no constant term; with an exception when the sides of the polygon are parallel to, or coincident with, the sides of the triangle.

Thus, corresponding to $\alpha\beta$ the equation is $ay^4 + bx^2y^3 = 0$, or $ay + bx^2 = 0$, and with this relation between x and y, every other term vanishes compared with the terms in α and β, when x and y are indefinitely great. Fig. 20.

Again, corresponding to $\beta\gamma$, we have the equation $bx^2y^3 + cx^3y^2 + dx^4y = 0$, or $by^2 + cxy + dx^2 = 0$, which represents two straight lines, and, with this relation at an infinite distance, every other term vanishes compared with any one of the three terms retained.

Again, taking the side $\gamma\delta$, the equation is $dx^4y + ex^2 = 0$, or $dx^2y + e = 0$, with which relation, x being infinite, and y indefinitely small, every other term will vanish compared with the terms retained, thus xy and x^2y^3 will be of the order $\frac{1}{x}$ and $\frac{1}{x^4}$ respectively.

Corresponding to $\delta\epsilon$, $ex + fy = 0$, and, with this relation, every other term vanishes when x and y are indefinitely small.

PLATE VI.

Lastly, $\epsilon\alpha$ gives $ay^2+fx=0$, which relation makes every other term vanish for points taken near the origin.

149. I think the following method of considering this triangle is much more convenient, both for placing the terms in any particular case, and for exhibiting the properties of the polygon, also, when necessary, in fixing more accurately the position of crosses which may correspond to fractional indices of x and y.

Instead of making squares, which act as cells in which the terms are placed, take a right-angled isosceles triangle, whose sides are in the directions Ox and Oy, measure equal distances along both sides, numbered, or supposed to be so, from 1 to n, and, through each such division, draw lines parallel to the sides, and terminated by the hypothenuse. Each point of intersection of such lines corresponds to a term of the complete general equation of the n^{th} degree.

The simplest way of drawing the figure is to divide the hypothenuse into as many equal parts as the degree of the equation, and to draw parallel lines from the points of division.

The equation given in the last article would be placed upon the triangle as in the figure. Fig. 21.

If a fractional index occurred in the equation, suppose $x^{\frac{3}{2}}y^{\frac{7}{3}}$ we should register its position by the intersection of two lines parallel to Oy, Ox bisecting 12 in Ox and trisecting 23 in Oy.

I shall adopt this form of the Analytical Triangle, using a small circle instead of a cross as being the most convenient mark to place in the intersection of two lines.

PROPERTIES OF THE ANALYTICAL TRIANGLE.

150. When all the terms of an equation of a curve are placed by circles upon the triangle, the following properties hold, with respect to any straight line which contains two or more of the circles.

i. If every term of the equation be rejected, except the terms which correspond to the circles which lie in a given

line, the resulting equation gives one or more constant values of the ratio $y^s : x^r$, the values of r and s depending only on the direction of the line, and being therefore the same for all parallel lines.

ii. If the straight line meet both sides of the triangle, or these produced beyond the hypothenuse, with the relation $y^s \propto x^r$, the terms of the original equation whose circles lie on the same side of the straight line as the right angle O, will be less, and those whose circles are on opposite sides will be greater, than any of the terms whose circles lie upon the line, when x and y are indefinitely great; and the reverse takes place when x and y are indefinitely small, in which case there is no circle at the right angle.

iii. When the line makes acute angles with both sides of the triangle, the relation is of the form $y^s x^r = \text{constant}$, and when y is indefinitely great and x indefinitely small, the terms whose circles are on the y side of the line are greater, those on the x side less, than any term whose circle is on the line, and *vice versâ* if x be great and y small.

iv. When the line is parallel to one of the sides, say to Ox, the resulting equation gives one or more straight lines which are parallel to Oy; and when y is indefinitely great each term whose circle is on the same side as O is less, and those whose circles are on the opposite side are greater than those whose circles are on the line.

v. When the line coincides with a side of the triangle, the solution of the resulting equation gives the points of intersection with the corresponding axis.

151. The truth of these propositions is easily seen by taking particular cases, but they may be shewn generally as follows:

Let r, s and r', s' be indices of any two terms whose circles lie on the line, these satisfy the linear equation $Ax + By = C$.

$$\therefore\ Ar + Bs = C,$$
$$Ar' + Bs' = C,$$
$$A(r' - r) + B(s' - s) = 0.$$

PLATE VI.

The equation resulting from the rejection of every term of the original equation, whose circle does not lie on the given line, is of the form

$$ax^r y^s + a'x^{r'}y^{s'} + \ldots = 0.$$

Dividing by $x^r y^s$ we have

$$a + a'x^{r'-r}y^{s'-s} + \ldots = 0,$$

$$\text{or } a + a'(x^{\frac{r'-r}{s'-s}}y)^{s'-s} + \ldots = 0,$$

$$\text{or } a + a'(x^{-\frac{B}{A}}y)^{s'-s} + \ldots = 0,$$

and the solution of this equation gives one or more constant values of $y : x^{\frac{B}{A}}$ which proves (i), since $\frac{B}{A}$ depends only on the direction of the line, and not on its position.

152. Employing this relation, every term is of the order $x^{r+\frac{sB}{A}} = x^{\frac{C}{A}}$ or $y^{\frac{C}{B}}$ dependent on the position of the line, and not on that of the circles which lie upon it.

If the line cut both sides of the triangle, or these produced, A and B are positive, and when x and y are very great, the terms, whose circles lie in a line parallel to the given line, the equation of which is $Ax + By = C'$, are greater or less than the terms retained, as C' is greater or less than C, *i.e.* as the circles lie on the same side of the line as O or on opposite sides; *vice versâ* if x and y can be very small. This proves (ii).

If the line make an acute angle with each side of the triangle, A and B have contrary signs; when C and A are positive and x very great, y is very small, and as C' is greater or less than C, the corresponding terms are greater or less than those retained. This proves (iii).

(iv) and (v) are evident.

153. By means of these propositions all the properties of the bounding polygon are immediately proved.

PLATE VI.

154. From the proof given above we may remark, that if any line give the first approximation to an asymptote, the terms to be taken into account for the second approximation are found by moving the line parallel to itself until it passes through another circle or set of circles, all of which correspond to the terms required to be taken into account.

155. I shall now shew the use of the triangle by applying it to assist in tracing the following curves:

(1) $$x^6 + 2a^2x^3y - b^3y^3 = 0.$$

Place the equation on the triangle, and draw the polygon, two sides of which correspond to branches through the origin and one to the infinite branch, viz. Fig. 22.

$$x^3 + 2a^3y = 0$$  ,

$$2a^2x^3 - b^3y^2 = 0$$  ,

and $x^3 - by = 0$  . Fig. 23.

(2) The next example is one which has already been considered in Art. 93, viz.

$$x^3y^2 + xy^5 - y^7 - x^7 = 0.$$

Placing the equation on the triangle we obtain the polygon $\alpha\beta\gamma\delta$. Fig. 24.

$\alpha\beta$ gives $y^2 - x^4 = 0$  ,

$\beta\gamma$ ...... $x^2 + y^3 = 0$  ,

$\gamma\delta$ ...... $x - y^2 = 0$  .

The first approximation to the asymptote given by $\delta\alpha$ is $y + x = 0$; the next approximation is found by taking into account the circle γ, which is first met by moving $\alpha\delta$ parallel to itself towards O; the resulting equation gives the rectilinear asymptote, viz.

$$y + x + \tfrac{1}{7} = 0.$$

PLATE VI.

The next approximation gives $y + x + \frac{1}{7} = \frac{5}{49x}$, taking into account the circle β.

To trace the curve, observe that

$$x = y = 1, \text{ or } -\tfrac{1}{2}.$$

Near $(1, 1)$, if $x = 1 + \xi$, $y = 1 + \eta$, the coefficients of $\eta = 0$; therefore, proceeding to η^2,

$$3\xi + 10\eta^2 = 0.$$

Near $(-\frac{1}{2}, -\frac{1}{2})$, if $x = -\frac{1}{2} + \xi$, $y = -\frac{1}{2} + \eta$,

$$\xi = 2\eta.$$

Again, $$x = -y = 1,$$

$$\text{near } (1, -1), \ 5\xi + 4\eta = 0.$$

To find the size of the smallest loop $x = -\frac{1}{4}y$ gives nearly $\frac{1}{4^3} + \frac{1}{4}y + y^2 = 0$, or two equal values of y, each $= -\frac{1}{8}$.

$$x = -\frac{1}{5}y, \text{ gives } \frac{1}{5^3} + \frac{1}{5}y + y^2 = 0;$$

$$\therefore \ y^2 + \frac{1}{5}y + \frac{1}{100} = \frac{1}{100} - \frac{4}{5}\frac{1}{100} = \frac{125}{250^2};$$

Fig. 25. $$\therefore \ y = \frac{\pm 11 - 25}{250} = -\frac{1}{18} \text{ or } -\frac{1}{7} \text{ nearly.}$$

A magnified figure of the shape near the origin is given on account of the difficulty of shewing the form of the

Fig. 26. small loop.

The continuity of the curve may be seen by commencing at the asymptote in $x'Oy$, passing through the origin along $y = x^2$, tracing the loop in xOy, passing through the origin along $y^2 = x$, tracing the small loop in xOy', through the origin along $x^2 + y^3 = 0$, then along the loop in $x'Oy'$, through the origin along $-y = x^2$, and so to the other end of the asymptote.

(3) $$x(x^2 - ay)^2 - y^5 = 0.$$

Placing on the triangle, we obtain the two forms at

Fig. 27. the origin

$$(ay - x^2)^2 = 0, \text{ and } a^2x = y^3.$$

. 5

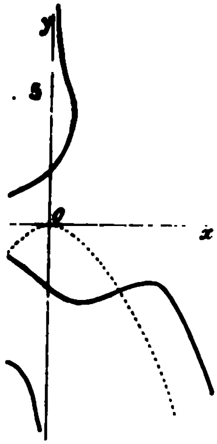


Fig 8

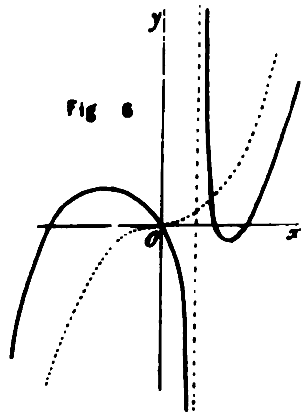


Fig 11

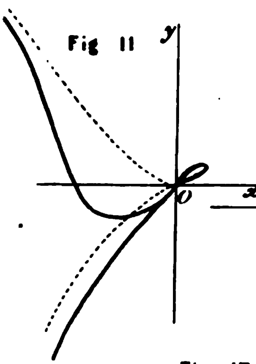


Fig. 12

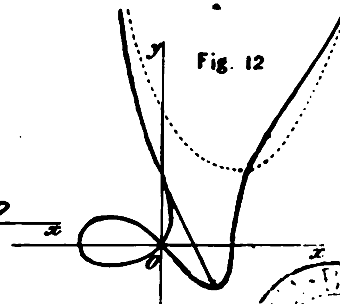


Fig. 17

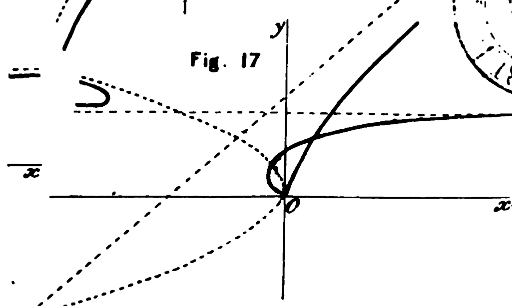


Fig. 22

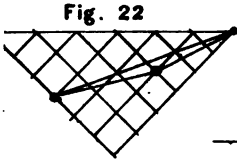


Fig. 27

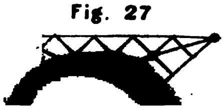
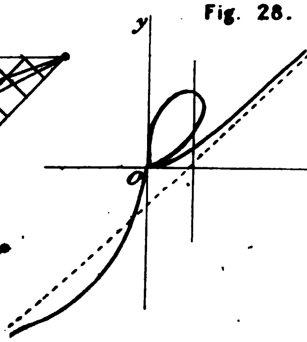


Fig. 28.



W. Metcalfe & Sons, Liths.

:t approximation to the first is PLATE VI.

$$ay = x^2 + \left(\frac{x^9}{a^5}\right)^{\frac{1}{2}}$$ .

mptote corresponding to $x^5 = y^5$ is

$$y = x - \frac{2a}{5}.$$

:t approximation is

$$y = x - \frac{2a}{5} + \frac{a^2}{25x}.$$

ring that the curve must cut the asymptote in
ıt $x'Oy'$ it is seen that the loop cannot cut it, for
would be four points of intersection at a finite
ısides two at an infinite distance.

ve is parallel to Ox at a point near $(\frac{1}{3}a, \frac{5}{8}a)$. Fig. 28.

$$(y^2 - x^2)^2 + 2axy^2 - 5ax^3 = 0.$$

PLATE VII.

the equation on the triangle, the side $\alpha\beta$ gives
t an infinite distance, the next approximation gives Fig. 1.

$3ax^3$, whence $y^2 = x^2 \pm \sqrt{(3a)}\, x^{\frac{3}{2}}$,

$$\text{or } y = \pm x \pm \tfrac{1}{2}\sqrt{(3ax)}.$$

ɘ $\alpha\delta$ gives near the origin $y^2 + 2ax = 0$. The side
$= \pm\sqrt{\tfrac{5}{2}}x$. The side $\gamma\beta$ gives, for the points of
ı with Ox, $x^3(x - 5a) = 0$, and for the next ap-
ı, $x^3(x - 5a) + 2axy^2 - 2x^2y^3 = 0$,

$$\text{or } y^3 = \tfrac{25}{8}(x - 5a).$$

t employing the triangle, we should naturally solve
n with respect to y, thus

$$y^2 - x^2 + ax = ax\left(\frac{3x}{a} + 1\right)^{\frac{1}{2}}.$$

$$\text{o}, \infty),\ y^2 = x^2 - ax \pm ax\sqrt{\frac{3x}{a}}\left\{1 + \frac{a}{6x}\right\}$$

$$c^2\left\{1 + \sqrt{\left(\frac{3a}{x}\right)} - \frac{a}{x} + \frac{a}{6x}\sqrt{\left(\frac{3a}{x}\right)} - \dots\right\};$$

PLATE VII.

$$\therefore\ y = \pm x\left\{1 + \frac{1}{2}\sqrt{\left(\frac{3a}{x}\right)} - \frac{1}{2}\frac{a}{x} - \frac{1}{8}\frac{3a}{x}\right\},$$

whence $(y \mp x \pm \frac{7}{8}a)^2 = \frac{3}{4}ax$ is the proper asymptote.

$$\text{Near } (0, 0),\quad y^2 = a \pm ax\left(1 + \frac{3x}{2a}\right) - ax + x^2,$$

$$\therefore\ y^2 = \tfrac{5}{2}x^2, \text{ or } -2ax.$$

$$\text{Near } (5a, 0),\quad -2x^2y^2 + 2axy^2 + x^3(x - 5a) = 0;$$

$$\therefore\ y^2 = \tfrac{25}{8}a(x - 5a).$$

Fig. 2. The curve is parallel to Ox at the point $(-\frac{1}{3}a, \pm\frac{2}{3}a)$.

(5) $x^6 - 2bx^3y^2 - 2abx^4 + b^2y^4 - 2ab^2xy^2 + a^2b^2x^2 = 0.$

Fig. 3. Place the equation on the triangle, the side $\alpha\beta$ contains three circles, giving $x^6 - 2bx^3y^2 + b^2y^4 = 0$, or $(x^3 - by^2)^2$, as a first approximation near (∞, ∞); moving the line parallel to itself, it passes two circles which belong to the next approximation, viz. $-2ab^4 - 2ab^2xy^2$, whence

$$(x^3 - by^2)^2 - 4abx^4 = 0,$$

$$\text{or } by^2 = x^3 \pm 2\sqrt{(ab)}\,x^2.$$

The side $\alpha\gamma$ contains three circles, giving as the form near the origin $(y^2 - ax)^2 = 0$, and moving the line parallel to itself, two terms $-2bx^3y^2 - 2abx^4$ are introduced, giving as the next approximation $b^2(y^2 - ax)^2 - \dfrac{4b}{a^3}y^8 = 0$,

$$\therefore\ ax = y^2 \pm \frac{2y^4}{(a^3b)^{\frac{1}{2}}}.$$

The third side $\beta\gamma$ gives the points of intersection with Ox, $x^6 - 2abx^4 + a^2b^2x^2 = 0$, and again, moving the line parallel to Ox, the next approximation is

$$x^2(x^2 - ab)^2 - 2bxy^2(x^2 + ab) = 0;$$

$$\therefore\ x^2 = ab \pm 2\sqrt[4]{(ab^3)}\,y,$$

$$\text{or } x = \pm\sqrt{(ab)} + \sqrt[4]{\left(\frac{b}{a}\right)}\cdot y.$$

Without using the triangle, since the equation may be written PLATE VII.

$$b^2y^4 - 2x(x^2 + ab)\,by^2 + x^2(x^2 + ab)^2 = 4abx^4,$$

$$\text{or } by^2 = x(x^2 + ab) \pm 2\sqrt{(ab)}\,x^2$$

$$= x\{x \pm \sqrt{(ab)}\}^2;$$

$$\text{near } \{\sqrt{(ab)}, 0\}, \quad y = \sqrt[4]{\left(\frac{a}{b}\right)}\cdot\xi,$$

$$\ldots\ldots (\infty, \infty), \quad by^2 = x^3 \pm 2\sqrt{(ab)}\,x^2.$$

The curve is parallel to Ox, or has a multiple point where

$$3x^2 - 1\sqrt{(abx)} + ab = 0,$$

$$\text{or } x^2 - \tfrac{4}{3}\sqrt{(ab)}\,x + \tfrac{4}{9}ab = \tfrac{1}{9}ab,$$

$$x = \frac{2 \pm 1}{3}\sqrt{(ab)} = \sqrt{(ab)}, \text{ or } \tfrac{1}{3}\sqrt{(ab)}.$$ Fig. 4.

(6) $$x^3y^4 + ax^2y^3 + bx^4y + cx + dy^3 = 0.$$

Placing the equation upon the triangle, the polygon gives for infinite branches the sides $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, and the equations derived from these are Fig. 5.

$$x^3y^2 + d = 0, \quad Oy \text{ being an asymptote},$$

$$y^3 + bx = 0,$$

$$\text{and } bx^3y + c = 0, \quad Ox \text{ being an asymptote}.$$

The form near the origin is given by the equation $cx + dy^2 = 0$, corresponding to the side $\delta\alpha$.

It is easily seen that the next approximation to the asymptote gives

$$bx = -y^3 + \frac{ab}{y},$$

so that the first approximation gives a proper asymptote.

The figures are drawn for the two cases of b, c, d, being all positive and all negative, and the double branches marked + and − are the positions which correspond to a, positive or negative. Figs. 6, 7.

It may be shewn readily that other combinations of signs give the same forms with relation to axes reversed, or turned through a right angle.

PLATE VII.

Note. The curve meets the curvilinear asymptotes $y^3 + bx = 0$, where $ax^2y^3 + cx + dy^2 = 0$, which gives for determining the values of y,

$$ay^9 - bcy^3 + b^2dy^2 = 0,$$

this equation has two roots equal to zero, and one root positive or negative, according as a, d have different or the same sign; the roots of $ay^7 - bcy + b^2d = 0$ are the common ordinates of curves

$$(ax - bc)\,y + b^2d = 0, \text{ and } x = y^6,$$

whence it appears that there may be one or three roots, if a, bc, d are of the same sign, or a, bc of the same, d of a contrary sign; only one positive root, if bc, d be of the same, a of a contrary sign; and only one negative root if a, d be of the same, and bc of a contrary sign.

This shews the effect of the sign of a, which does not materially affect the general shape of the curve, as might be expected from the interior position of the corresponding circle in the triangle.

(7) $$a^3y^5 + x^2y^6 - x^5y^3 + ax^6y - a^6x^2 = 0.$$

Fig. 8. Placing in the triangle the four sides $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\delta\varepsilon$, give

$a^3 + x^2y = 0$, Oy being an asymptote,

$y^3 - x^3 = 0$,

$y^2 - ax = 0$,

$x^4y - a^5 = 0$, Ox being an asymptote.

The side $\varepsilon\alpha$ gives $y^5 - a^3x^2 = 0$ for the shape at the origin.

Moving $\beta\gamma$ to pass through δ, the second approximation gives $y - x + \frac{1}{3}a = 0$.

Moving $\gamma\delta$ parallel to itself, it passes next through β, and the next approximation to the parabolic asymptote is

$$y^2 - ax = \frac{y^5}{x^3} = \frac{a^3}{y},$$

shewing that the parabola is a proper asymptote.

The parabolic asymptote cuts the curve where

$$y^6 + a^5y - a^6 = 0,$$

PLATE VII.

and the values of y are the common ordinates of

$$a^2x = y^3, \text{ and } x^2 + a(y-a) = 0.$$

The next approximation to the rectilinear asymptote may be found from

$$y^3 - x^3 + a\frac{x^4}{y^2},$$

$$\begin{aligned} y &= x\left(1 - a\frac{x}{y^2}\right)^{\frac{1}{3}} \\ &= x\left(1 - \frac{1}{3}\frac{ax}{y^2} - \frac{1}{9}\frac{a^2x^2}{y^4}\right) \\ &= x\left\{1 - \frac{a}{3x}\left(1 + \frac{2a}{3x}\right) - \frac{1}{9}\frac{a^2}{x^2}\right\} \\ &= x - \frac{a}{3} - \frac{a^2}{3x}. \end{aligned}$$

Fig. 9.

(8) $(y-a)^3x^4 - 2a^4(y-a)x^2 + a^4(x-a)y^2 = 0.$

Placing on the triangle, one side of the polygon gives $y^3x^4 - a^5y^2 = 0$, or $yx^4 = a^5$, the axis of y being an asymptote; another side gives $(y-a)^3x^4 = 0$, if this side be moved parallel to itself, the next terms are $-2a^2(y-a)x^2$, the next approximation would make $(y-a)x$ finite, so that $(y-a)x^2$ is of the order x when x is infinite, to obtain the values of $y-a$ we must therefore include the next term a^4xy^2, the resulting equation being Fig. 10.

$$(y-a)^3x^3 - 2a^4(y-a)x + a^6 = 0,$$

which gives three finite values for $(y-a)x$, viz. a^2 and $\frac{1}{2}(\pm\sqrt{5} - 1)a^2$.

The shape at the origin of the original axes is given by the lower side of the polygon, viz.:

$$2x^2 - y^2 = 0.$$

The curve cuts the axis of x, where $x^2 = 2a^2$, and the line $y = a$, where $x = a$;

$$\text{near } (\pm a\sqrt{2}, 0), \text{ if } x = \pm a\sqrt{2} + \xi, \ \pm\sqrt{2}\,y = \xi;$$

$$\text{near } (a, a), \ 2\eta = \xi.$$

PLATE VII.

To assist in tracing the curve we observe that

Fig. 11. $$x = y = \tfrac{1}{2}(\pm\sqrt{5} + 1)\,a.$$

Note. The triangle is employed for the purpose of approximating, by writing y for $y - a$, and placing the resulting equation on the triangle,

Fig. 12. $$y^3x^4 - 2a^4yx^2 + a^4(x - a)(y + a)^2 = 0.$$

The corresponding side of the polygon gives

$$y^3x^4 - 2a^4yx^2 + a^6x = 0,$$

$$\text{or } yx = a^2 \text{ and } \frac{\pm\sqrt{5} - 1}{2}\,a^2.$$

(9) $$x(x^2 - ay)^2 = y^4(y - c).$$

Fig. 13. Placed on the triangle, the polygon gives, at an infinite distance, $x^5 - y^5 = 0$; and, moving the side parallel to itself, two more terms are introduced, giving for the asymptote

$$y - x = \tfrac{1}{5}(c - 2a),$$

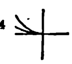
in the next approximation the term $\dfrac{(a - c)^2 + c^2}{25x}$ is added.

The forms near the origin are

$$a^2x + cy^2 = 0$$  ,

$$\text{and } (x^2 - ay)^2 = 0,$$

the next approximation being

$$(x^2 - ay)^2 = -\frac{c}{x}\,y^4$$  .

When $y = c$, $x = 0$, or $\pm\sqrt{(ac)}$;

near $(0, c)$, $a^2x = c^2\eta$;

...... $\{\pm\sqrt{(ac)}, c\}$, $\pm\sqrt{(ac)}\,.\,4ac\xi^2 = c^4y$.

The difficulty in tracing this curve consists in determining whether the branch through $(0, c)$ joins the branch $a^2x + cy^2 = 0$, or the branch through $\{-\sqrt{(ac)}, c\}$.

It will illustrate the artifices which may be used in such cases, if we examine the number of points where the curve is parallel to the axis of x, and afterwards the points where the tangent $a^2x = c^2(y - c)$ meets the curve.

PLATE VII.

The curve is parallel to Ox, where

$$(x^2 - ay)^2 + 4x^2(x^2 - ay) = 0,$$

$$\text{or } x = 0, \ x^2 - ay, \text{ and } 5x^2 = ay;$$

the first two have been considered.

$$\text{If } 5x^2 = ay, \ x.(4ay)^2 = 25y^4(y - c);$$

$$\therefore \ 16a^2x = 25y^2(y - c),$$

$$\text{or } 5^5y^3(y - c)^2 = 4^4a^5,$$

$$\text{or } y^3(y - c)^2 = \tfrac{1}{12}a^5 \text{ nearly.}$$

The values of y are the common ordinates of the curves

$$y(y - c) = \tfrac{1}{4}ax, \text{ and } yx^2 = \tfrac{4}{5}a^3.$$

The two curves are drawn for the case of $c = 2a$, and $c = a$, and they shew that there are three values of y in the first case, and only one in the second. Thus, when $c = 2a$, the loop is found by joining the ends of the ramphoid cusp, and, when $c = a$, by joining one branch of the cusp with the parabolic branch through the origin. Figs. 14, 15.

The other plan, of examining the point where the tangent $a^2x = c^2(y - c)$ meets the curve, leads to some elegant results.

We may notice that the next approximation gives

$$a^2x = c^2\eta + 2c\eta^2;$$

$$\therefore \ c(x^2 - ay) = \pm ay^2,$$

$$\text{or } \frac{x^2}{a} = \pm\frac{y^2}{c} + y.$$

The loci of these equations are a hyperbola, and an ellipse whose transverse axis is c, and the ratio of the axes is $a^{\frac{1}{2}} : c^{\frac{1}{2}}$, the positions are given in the figures for $c = 2a$, and $c = a$, in the latter case the loci being a circle and rectangular hyperbola; the points of intersection are $RAPQ$ for the first, R being at an infinite distance in the second. Figs. 16, 17.

The figure of the curve is drawn for the same relations between c and a. The letters $APQR$ correspond to the same letters in figures 16 and 17. Figs. 18, 19.

PLATE VII.

(10) $6x^7 - 2x^5y^2 - a^3x^2y^2 + 4a^3x^3y + 2a^5x^2 - 3a^5xy + a^5y^2 = 0.$

Fig. 20. Placed on the triangle, the polygon gives

near (∞, ∞), $3x^2 - y^2 = 0$, two asymptotes;

for asymptotes parallel to Ox,

$$2x^5 + a^3x^2 - a^5 = 0,$$

the solution of which is the common abscissa of

$$x^3 = a^2y, \text{ and } (2y + a)x^2 = a^3;$$

$$\text{near } (0, 0), \ 2x^2 - 3xy + y^2 = 0,$$

$$\text{or } y = x, \text{ and } 2x;$$

$$\text{and for } y = 0, \ 3x^5 + a^5 = 0.$$

It can be shewn that the curve lies on the y side of both asymptotes, $y = \pm x\sqrt{3}$, and cuts each at two points, for the upper sign less, and for the lower greater than a, also that it cuts the asymptote parallel to Oy at a considerable distance on the positive side.

Fig. 21. The figure can be traced continuously.

156. It will have been seen by the illustrations which have been given, that with equations of high degrees, the use of the Analytical Triangle is almost indispensable. But at the end of this work, in which I propose to shew how the inverse process, of finding the equation of a curve when the curve is given, can be performed, it will, I think be seen, that the value of the triangle is indefinitely increased; at all events, the new use which I have made of the triangle in this inverse process, will, perhaps, excuse me for having laid so much stress upon its properties in the present chapter.

IX.

Employ the Analytical Triangle to find the forms of the branches of the following curves near the origin, supposing it a point in the curve, and at an infinite distance when the curve has asymptotic branches; and draw the curves.

PLATE VII.

(1) $a^2y^3 - bx^3y + x^2 = 0$, or a^5.

(2) $y^4 - 2x^2y^2 - 3x^4 + 8ax^2y^2 - 16ax^3 = 0$.

Find the points where it meets the asymptotes, and compare the curvatures at the points where it meets Ox.

(3) $x^4 + y^4 - (cy + ax)\,x^2 = 0$.

(4) $x\,(x^2 - ay)\,(x^3 - by) + cy^4 - y^5 = 0$.

(5) $x^6 + y^6 - ay^5 + a\,(ay - cx)\,x^3 = 0$.

Trace the curve, and shew that if $c = 0$, a tangent from the origin to the curve is inclined to Ox at an angle $\tan^{-1}\frac{3}{2}$ nearly.

(6) $x^6 - x^4y^2 - 2xy^3 + 3y - x^2 = 0$.

(7) $x^6 + x^4y^2 - 2x^2y + y^2 - 2y - 2x^2 + 1 = 0$.

(8) $x^5y - ax^2y^3 + b^2y^4 + a^4xy - b^4x^2 = 0$.

(9) $a^2x^6 - 2a^2bx^4y + a^2b^2x^2y^2 - 2ab^2xy^4 + b^2y^6 = 0$.

(10) $x^6y^2 - 2a^2bx^4y + a^4b^2x^2 - 2a^4b^2xy + a^4b^2y^2 = 0$.

CHAPTER X.

SINGULAR POINTS. DIVISION INTO COMPARTMENTS. SPECIAL CURVE OF THE FOURTH DEGREE.

PLATE VII.

157. In the preceding chapters I have been endeavouring to make clear to the student processes which will enable him to determine, as exactly as he pleases, the form of a curve in the neighbourhood of any point, whether at a finite or infinite distance, which he may have occasion to consider.

I have also shewn how a certain polygon on the Analytical Triangle supplies a test whether all the branches have been considered, which may exist at an infinite distance, or pass through the origin.

158. In order to trace a curve we must generally find such a finite number of points whose coordinates satisfy the equation, that, having the tangents to the curve at those points, and the direction of the deflection of the curve from the tangents at the most important points, there shall be only one way of joining the corresponding small elements, which will not be inconsistent with the degree of the equation and the laws of continuity, and other properties obvious from the equation, such as symmetry, &c.

159. I have said *generally*, because it may be possible to obtain from the equation some geometrical construction which will amount practically to giving an infinite number of points; as, for example, $(x-a)^2+(y-b)^2=c^2$ represents that the curve is the locus of a point whose distance from (a, b) is always c; and $y^2=4ax$, is the same as

$$y^2+(x-a)^2=(x+a)^2,$$

PLATE VII.

which expresses that the curve is the locus of a point whose distance from $(a, 0)$ is equal to the distance from the line $x + a = 0$.

160. In the case of equations which can be solved with respect to either of the coordinates, so as to express it explicitly in terms of the other, there is no limit to the number of points which may be found, but the form of the result will, in most cases, suggest the peculiar points which will most easily lead to the form of the curve, so as to limit the number of points necessary to be examined.

But in the case of equations which cannot be so dealt with, it will be found that, when we have constructed all the forms of the curve in the neighbourhood of points which readily present themselves to be examined, there will be many ways in which these elements can be connected by curved lines, none of which, on the face of it, contradict the properties of the equation.

Moreover, supposing that the elements so determined could be joined in only one way, some outlying portion of the curve, such as an oval, by the existence of which no law of continuity would be broken, may have been omitted entirely, if it so happened that none of the elements already found were part of it.

161. I shall now endeavour to supply some means of meeting these difficulties, not professing that they will all admit of practical application in every case, but suggesting them as methods to be tried, before the curve is given up in despair.

162. The most important points which give character to a curve next to the infinite branches are multiple points of all sorts, cusps, and conjugate points; it is therefore necessary to call special attention to the method of determining their position when they exist, although it has already been incidentally mentioned in a note, p. 92.

. the conditions of the ...ained, is that every straight ...hese points, whatever be its ...curve in at least two coincident ...en to be the case for multiple points ...ing the straight line a small distance ... the points which were coincident in the ...arate, and become distinct, as the line moves ...sition.

SINGUL...

PLATE VII.

15... to

...ate point may be considered as the limit of a small oval or closed curve, by which supposition ...perty holds also in this case.

But before giving the analysis for these points I must direct attention to peculiar points which occur in some curves, for which the property spoken of holds as well as for multiple points and cusps, although nothing in the form of the curve accounts directly for it.

164. If we take a curve for which a branch through the origin is $y^4 = ax^3$  , a straight line $y = mx$ drawn in any direction defined by m meets the curve in three points at least at the origin.

The existence of these points which do not appear in the form of the curve, but only as an algebraical result, may be accounted for by considering the curve as degenerated from a more complete curve; this degeneration has been observed by E. Walker and Walton in the *Quarterly Journal*.

165. As an example of this degeneration, we may consider the curve whose equation is $y^4 = ax^3$, as the limit of one whose equation is

$$y(y-\alpha)(y-\beta)(y-\gamma) = ax(x-\delta)(x-\varepsilon),$$

when the constants denoted by Greek letters are made indefinitely small.

In order to trace this curve, where α, β, ... are real,

PLATE VII.

points are seen through which the curve passes for all ons of values of the constants.

nts where the curve is parallel to Ox would be lly by two real values of x, to each of which ond four, two, or no values of y; in particular se points, or some of them, might be replaced by ple points.

The curve is parallel to Oy at points given by a cubic equation, whose solution may give one or three real values, to each of which may correspond one or three values of x.

The asymptotic branch is $y^4 = ax^3$. The forms of the curve which satisfy these conditions necessarily involve many points of inflexion, and care must be taken that no tangent at any point of inflexion cuts the curve again in two points, which would require five points of intersection.

The figures given represent various forms which might be assumed by the curve, all of which may, by gradual changes of the constants, be made to pass from one to another; while at every change of general form, the parts of the curve which bend towards each other, produce multiple points. Fig. 22.

As the constants diminish and ultimately vanish, the four values of y, when $x = 0$, and the three values of x, when $y = 0$, are accounted for; also the three points in which the curve is met by any line drawn through the origin are explained, the fourth remaining point being the point of intersection with the parabolic asymptote.

166. If two values of x corresponding to $y = 0$ are impossible, as when the equation of the curve is of the form

$$y(y-\alpha)(y-\beta)(y-\gamma) = ax\{(x-\delta)^2 + \varepsilon^2\},$$

it is easily seen that the equation for determining the values of x where the curve is parallel to Ox, has real, equal, or impossible roots, as δ is $>$, $=$, or $< \sqrt{3}\varepsilon$; and as the corresponding values of y may have four real values, two, or none, the curve may assume any of the forms in the figures, the upper belonging to $\delta > \sqrt{3}\varepsilon$, the lower to $\delta = \sqrt{3}\varepsilon$ and $\delta < \sqrt{3}\varepsilon$. Fig. 23.

PLATE VII.

167. As another example of the fact that a curve of the form $y^4 = ax^3$ is cut by a line drawn through the origin in these points, take the curve

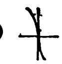
$$y^6 + x^6 - ay^5 + a(ay - cx)x^3 = 0.$$

There is no infinite branch, and the forms near the origin are given by

$$ay - cx = 0, \text{ and } y^4 = ax^3.$$

The next approximation to $ay = cx$ is

$$ay - cx = \frac{c^5}{a^5}x^2;$$

$$\text{near } (0, a), \qquad a^2\eta + x^3 = 0$$  ,

$$\ldots\ldots \{\pm\sqrt{(ac)}, 0\}, \quad 2c\xi + ay = 0,$$

$ay - cx = 0$ meets the curve where

$$\left(1 + \frac{c^6}{a^6}\right)x = \frac{c^5}{a^4}.$$

Figs. 24,25. The figures are drawn for $c = 2a$ and a.

The six points of intersection of any straight line through the origin are accounted for by the three points in which it meets $y^4 = ax^3$.

SINGULAR POINTS.

168. The singular points, which are now to be considered, are those points, through any of which, if a straight line be drawn *in any direction*, it will have at least two of the points of intersection with the curve coincident with that point.

Hence, if (α, β) be a singular point of the curve, whose rationalized equation is $f(x, y) = 0$, and the origin be transferred to (α, β) by writing for x and y, $\alpha + \xi$ and $\beta + \eta$, $f(\alpha + \xi, \beta + \eta)$ can contain no term of the order ξ or η. If therefore $\phi(\alpha, \beta)$ and $\psi(\alpha, \beta)$ are the coefficients of ξ and η respectively in the expansion of $f(\alpha + \xi, \beta + \eta)$, α, β must satisfy simultaneously the three equations

$$f(\alpha, \beta) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{i},$$

$$\phi(\alpha, \beta) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{ii},$$

$$\psi(\alpha, \beta) = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{iii}.$$

PLATE VII.

When these solutions are found, the origin of coordinates being transferred to the corresponding points, the form of curve is known by the methods of the preceding chapters.

The solutions of (i) and (ii), which do not at the same time satisfy (iii), give points where the curve is parallel to Ox, the approximate equations being of the form $\eta = A\xi^m$. Similarly, the solutions of (i) and (iii), which do not satisfy (ii), give points where the curve is parallel to Oy.

169. If $\alpha^r\beta^s$ be any term of (i), $r\alpha^{r-1}\beta^s$ and $s\alpha^r\beta^{s-1}$ will be the corresponding terms of (ii) and (iii), multiplying (ii) and (iii) by α and β, and adding, the corresponding term will be $(r+s)\,\alpha^r\beta^s$.

Therefore, arranging the equation (i) in the form of homogeneous functions of α, β, viz.

$$f(\alpha, \beta) \equiv u_n + u_{n-1} + \ldots + u_0 = 0,$$

(ii) and (iii) supply the equation

$$nu_n + (n-1)\,u_{n-1} + (n-2)\,u_{n-2} + \ldots + u_1 = 0;$$

$$\therefore\ u_{n-1} + 2n_{n-2} + \ldots + (n-1)\,u_1 + nu_0 = 0$$

is an equation which may be used instead of, or with the three equations given above.

170. The following examples will shew how the above equations are to be applied:

(1) $$(y^2 - a^2)^3 + x^4\,(2x + 3a)^2 = 0.$$

If (x, y) be a multiple point, writing $y + \eta$ for y,

$$(y^2 - a^2 + 2y\eta + \eta^2)^3 - x^4\,(2x + 3a)^2 = 0;$$

therefore, equating the coefficient of η to zero,

$$6y\,(y^2 - a^2)^2 = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ \text{i.}$$

Again, writing $x + \xi$ for x,

$$4x^3\,(2x + 3a)^2 + x^4.4\,(2x + 3a) = 0 \ \ldots\ldots\ldots\ \text{ii.}$$

The solutions of these equations are

$$y = 0 \text{ and } \pm a,$$

$$x = 0, \ -\tfrac{3}{2}a, \text{ and } -a,$$

PLATE VII. of these solutions $(0, \pm a)$, $(-\frac{4}{3}a, \pm a)$, and $(-a, 0)$ satisfy the given equations.

The direction of the branches at these points are given,

$$\text{near } (0, \pm a), \text{ by } (2a\eta)^3 + 9a^2x^4 = 0,$$
$$\text{or } 8a\eta^3 + 9x^4 = 0;$$
$$\text{near } (-\tfrac{4}{3}a, \pm a), \text{ by } (2a\eta)^3 + (\tfrac{4}{3}a)^4 . 4\xi^2 = 0,$$
$$\text{or } 32\eta^3 + 81a\xi^2 = 0;$$
$$\text{near } (-a, 0), \text{ by } 3a^4y^2 - a^6 + (a-\xi)^4(a+2\xi)^2 = 0,$$
$$\text{or } 3a^4y^2 - a^6 + \{(a^2 - 2a\xi + 2\xi^2)(a+2\xi)\}^2 = 0,$$
$$\text{or } 3a^4y^2 - a^6 + \{a(a^2 - 4\xi^2) + a\xi^2\}^2 = 0,$$
$$\therefore\ 3a^4y^2 - 6a^4\xi^2 = 0, \text{ or } y^2 = 2\xi^2.$$

Hence the two latter points give a ceratoid cusp and a double point respectively.

The other points of the curve which satisfy the equation (i) are points at which the curve is parallel to Oy.

Thus if $y = 0$, $x^2(2x + 3a) = \pm a^3$.

Taking the upper sign $(x+a)^2(2x-a) = 0$.

Taking the lower sign $2x^3 + 3ax^2 + a^3 = 0$,

whose solution is the common abscissa of the curves

$$ay = x^2, \text{ and } (2x + 3a)y + a^2 = 0,$$

giving only one value of x, viz. $\frac{3}{5}a$, nearly.

$x = -a$, which satisfies (ii), gives

$$y^4 - 3a^2y^2 + 3a^4 = 0,$$

and therefore this value gives no point at which the curve

Fig. 26. is parallel to Ox.

(2) $(y^2 - x^2)(x-1)(x-\frac{3}{2}) = 2(y^2 + x^2 - 2x)^2$.

If $x + \xi$, $y + \eta$ be written for x and y, at a singular point the coefficient of ξ and η vanish;

$$\therefore\ y^2(2x - \tfrac{5}{2}) - 4x^3 + \tfrac{15}{2}x^2 - 3x = 8(x-1)(y^2 + x^2 - 2x) \ldots\text{i},$$
$$\text{and } y(x-1)(x-\tfrac{3}{2}) = 4y(y^2 + x^2 - 2x) \ldots\ldots\ldots\text{ii}.$$

The solutions of (ii) and the equation of the curve which

PLATE VII.

give the points where the curve is parallel to Oy, as well as the singular points, are

(*a*) $y=0$, $x=0$, or $3x^2-\frac{24}{2}x+\frac{19}{2}=0$, of which $(0, 0)$ is the only real solution, which is a singular point, since it also satisfies (i):

(*b*) $x=1$, $y^2=1$, which satisfy (i) and therefore give two more singular points:

(*c*) $x=\frac{3}{2}$, $y^2=\frac{3}{4}$, which do not satisfy (i) but give points where the curve is parallel to Oy:

(*d*) $$\text{and } 4(y^2-x^2)=2(y^2+x^2-2x);$$
$$\text{or } y^2=3x^2-2x;$$
$$\therefore (x-1)(x-\tfrac{3}{2})=4(4x^2-4x);$$
$$\therefore x-\tfrac{3}{2}=16x, \text{ or } x=-\tfrac{1}{10},\ y^2=\tfrac{23}{100},$$

which give two other points where the curve is parallel to Oy.

The points where the curve is parallel to Oy could be most easily found by solving the equation

$$(y^2+x^2-2x)^2-\tfrac{1}{2}(y^2+x^2-2x)(x-1)(x-\tfrac{3}{2})$$
$$=x(x-1)^2(x-\tfrac{3}{2});$$
$$\therefore \{y^2+x^2-2x-\tfrac{1}{4}(x-1)(x-\tfrac{3}{2})\}^2$$
$$=\tfrac{1}{16}(x-1)^2(x-\tfrac{3}{2})(x-\tfrac{3}{2}-16x);$$

whence $x=\frac{3}{2}$ and $x=-\frac{1}{10}$ each give equal values of y^2.

Also, where $x=0$, $\frac{3}{2}y^2=2y^4$; $\therefore y^2=\frac{3}{4}$ and 0.

$$\text{Near } (0, 0),\ \tfrac{3}{2}(y^2-x^2)=8x^2;$$
$$\therefore y^2=\tfrac{19}{3}x^2,\ y=\pm\tfrac{14}{5}x \text{ nearly}.$$
$$\text{Near } (1, 1),\ -(\eta-\xi)\xi=8\eta^2;$$
$$\therefore \xi^2-\eta\xi+\tfrac{1}{4}\eta^2=\tfrac{33}{4}\eta^2, \text{ or } \xi=\tfrac{1}{2}(1\pm\sqrt{33})\eta.$$ Fig. 27.

(3) $$xy^2+2a^2y-ax^2-3a^2x-3a^3=0.$$

Writing $x+\xi$, $y+\eta$, for x and y, the coefficients of ξ and η equated to zero, give

$$y^2-2ax-3a^2=0 \text{ .... i},$$

and $$2xy+2a^2=0 \text{ .... ii}.$$

PLATE VII.

These and the equations of the curve give $x=-a$, $y=a$. The shape near $(-a, a)$ is given by

$$-a\eta^2+2a\xi\eta-a\xi^2+\xi\eta^2=0;$$

$$\therefore (\eta-\xi)^2=\frac{\xi^3}{a}.$$

Near $(0, \infty)$, $x=-\dfrac{2a^2}{y}$;

...... (∞, ∞), $y^2=ax$,

...... $(0, \frac{3}{2}a)$, $(\frac{9}{4}a^2-3a^2)\,x+2a^2\eta=0$,

Fig. 28. or $8\eta=3x$.

(4) $(y^2-x^2)^2+6axy^2-7ax^3-4a^2y^2+18a^2x^2-20a^3x+8a^4=0$.

The equations for determining singular points are

$$-4x\,(y^2-x^2)+6ay^2-21ax^2+36a^2x-20a^3=0 \;\ldots\ldots\; \text{i},$$

and
$$4y\,(y^2-x^2)+12axy-8a^2y=0 \;\ldots\ldots\ldots\ldots\; \text{ii},$$

or $y=0$, and $y^2-x^2+3ax-2a^2=0$;

where $y=0$, $x^4-7ax^3+18a^2x^2-20a^3x+8a^4=0$,

or $(x-a)\,(x^3-6ax^2+12a^2x-8a^3)$,

or $(x-a)\,(x-2a)\,(x^2-4ax+4a^2)$,

or $(x-a)\,(x-2a)^3$.

The equation might have been written

$$y^4-2y^2\,(x-a)\,(x-2a)+(x-a)\,(x-2a)^3=0,$$

or $\{y^2-(x-a)\,(x-2a)\}^2=(x-a)\,(x-2a)^2\,a$.

Near $(a, 0)$, $y^4-a^3\xi=0$,

...... $(2a, 0)$, $2y^2=\xi^2$, and $y^2-2a\xi=0$,

...... (∞, ∞), $(y\mp x)^2=\frac{1}{4}ax$.

The proper parabolic asymptotes, as well as the side on which the curve lies, may be found by expanding y as far as terms in $x^{-\frac{1}{2}}$,

$$(y^2-x^2+3ax-2a^2)^2=a\,(x-a)\,(x-2a)^2;$$

$$\therefore\; y^2=x^2-3ax+2a^2+(ax)^{\frac{1}{2}}\,(x-2a)\left(1-\frac{1}{2}\frac{a}{x}-\frac{1}{8}\frac{a^2}{x^2}\right);$$

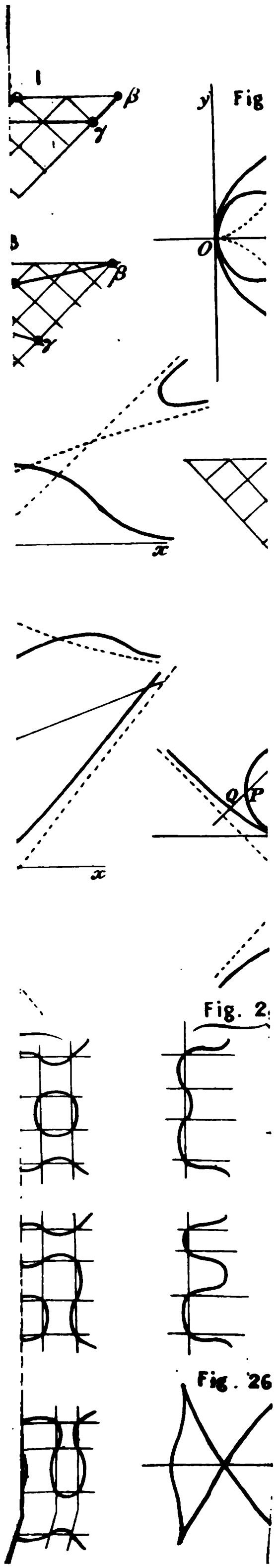

β
γ
y
Fig
O
x
Q
P
Fig. 2
Fig. 26

PLATE VII.

$$\therefore\ y = x\left\{1 + \left(\frac{a}{x}\right)^{\frac{1}{2}} - \frac{3a}{x} - \frac{5}{2}\left(\frac{a}{x}\right)^{\frac{3}{2}}\right\}^{\frac{1}{2}}$$

$$= x\left\{1 + \frac{1}{2}\left(\frac{a}{x}\right)^{\frac{1}{2}} - \frac{13a}{8x} - \frac{7}{16}\left(\frac{a}{x}\right)^{\frac{3}{2}}\right\};$$

$$\therefore\ (y - x + \tfrac{13}{8}a)^2 = \tfrac{1}{4}ax$$

is a proper asymptote, the curve lying on the concave side of both branches. Fig. 29.

(5) $27y^4 - 9y^2(x^2 + 14ax + a^2) + 32a(a + x)^3 = 0.$

The conditions for a singular point are

$$2y\{54y^2 - 9(x^2 + 14ax + a^2)\} = 0 \ldots\ldots\ldots\ldots\text{i}$$

$$-9y^2(2x + 14a) + 3.32a(a + x)^2 = 0 \ldots\ldots\ldots\text{ii},$$

$y = 0$, $x = -a$ satisfy the three equations, and, from (i) and (ii),

$$(x^2 + 14ax + a^2)(x + 7a) = 32a(x + a)^2,$$

$$(x^2 + 14ax + a^2)(x - a) = 8a(4x^2 + 8ax + 4a^2 - x^2 - 14ax - a^2)$$

$$= 24a(x^2 - 2ax + a^2);$$

$$\therefore\ x = a, \text{ and } x^2 - 10ax + 25a^2 = 0,$$

$$y = \pm\sqrt{\tfrac{8}{3}}a, \text{ and } \pm 4a,$$

of which $(5a, \pm 4a)$ satisfy the equation of the curve, which may therefore be written

$$3(6y^2 - x^2 - 14ax - a^2)^2 - (x - 5a)^3(3x + a) = 0,$$

hence, if $\quad x = 5a + \xi,\ y = \pm 4a + \eta,$

$$3(\pm 6.8a\eta - 24\xi)^2 - 16a\xi^3 = 0, \text{ neglecting } \xi^4,$$

$$\text{or } \pm\eta = \tfrac{1}{2}\xi + \tfrac{1}{12}(3a)^{-\frac{1}{2}}\xi^{\frac{3}{2}},$$

$$\text{near } (-a, 0),\ 27ay^2 + 8\xi^3 = 0.$$

For the rectilinear asymptotes the first approximation is $3y^2 = x^2$, the second is obtained from

$$9(3y^2 - x^2)y^2 - 9.14axy^2 + 32ax^3 = 0;$$

$$\therefore\ \pm\sqrt{3}y - x - \frac{3.14ax^3 - 32ax^3}{3x^2.2x} = 0,$$

and the equations of the asymptotes are

$$\pm\sqrt{3}y - x - \tfrac{5}{3}a = 0.$$

PLATE VIII.

The parabolic asymptote is

$$-9y^2 + 32ax = 0.$$

The proper asymptote being

Fig. 1.

$$y^2 = \frac{32a}{9}(x - \tfrac{1}{8}a).$$

Note. This tricusped curve is the locus of a point through which if the three maximum or minimum chords be drawn in a parabola $y^2 = 4ax$, two of the three chords coincide.*

The equations may be written

$$(x - 5a)^{-\frac{1}{2}} + (4x - 6y + 4a)^{-\frac{1}{2}} + (4x + 6y + 4a)^{-\frac{1}{2}} = 0.$$

DIVISION INTO COMPARTMENTS.

171. A great assistance in selecting the proper mode of making the junction of isolated parts of a curve, which may have been already determined, is acquired by observing that, if $f(x, y) = 0$ be an equation which does not represent separate curves, an even number of which coincide, when a variable point moves so as to cross the curve, every time that it so crosses, the value of the function $f(x, y)$, obtained by substituting the coordinates of the point for x and y, will change sign. This does not require any proof.

172. We are thus enabled to name certain compartments within which the curve cannot lie, when one point has been determined, and the general form of the curves, with equations of high degrees, can be frequently obtained with very little calculation.

Any of the examples which have been given of homogeneous curves will serve to illustrate this in a simple way.

The following curves will shew how to choose the compartments, and how to obtain the direction of the curve in passing from one to another compartment; in which it will be seen that by rearranging the equation of the curve in different forms, different sets of compartments may be

* Given to me by J. Wolstenholme.

PLATE VIII.

mapped down, and so assistance given in determining doubtful cases.

(1) $$xy(x^2-y^2)+ra^2(x^2+y^2-a^2)=0.$$

The four asymptotes are represented in the figure, viz. yOy', xOx', AOA', BOB', and the circle $x^2+y^2-a^2=0$ by $ab\ldots d'$. Fig. 2.

The intersections of the asymptotes with the circle, a, b, &c. are all points in the curve.

The tangent at $(0, a)$ is $-a^2x+ra^2.2a\eta=0$; if, therefore, r be positive, this shews two compartments dOc and $ycbA$ in which the curve must lie, and since none of the five lines mentioned above can be crossed separately without a change of sign, the compartments bOc, dOa', $b'Oc'$, $d'Oa$, and $xabA$, $ycdB$, $x'a'b'A'$, $y'c'd'B'$ must be empty; whence the curve is readily drawn.

The figure represents the three principal forms which the curve can assume according as $r <, =,$ or >1, denoted by the letters β, α, γ respectively.

To find the value of r, in order that there may be a multiple point, the following three equations must be satisfied simultaneously:

$$3yx^2-y^3+2a^2rx=0\ldots\ldots\ldots\ldots\ldots\ldots\text{i},$$

$$3xy^2-x^3-2a^2ry=0\ldots\ldots\ldots\ldots\ldots\ldots\text{ii},$$

$$\text{and } 2(x^2+y^2)-4a^2=0, \text{ Art. 169}, \ldots\ldots\text{iii}.$$

Multiply (i) and (ii) by y and x, and add,

$$6x^2y^2-y^4-x^4=0;$$

$$\therefore\ y^4-6x^2y^2+9x^4=8x^4,$$

$$y^2=(3\pm2\sqrt{2})\,x^2=(\sqrt{2}\pm1)^2x^2;$$

$$\therefore\ y=(\sqrt{2}\pm1)\,x;$$

$$\therefore, \text{ by (iii)}, \ (4\pm2\sqrt{2})\,x^2=2a^2,$$

$$3x^2-y^2=\mp2\sqrt{2}x^2=\mp(\sqrt{2}\mp1)\,2a^2;$$

$$\therefore, \text{ by (i)}, \ \mp2a^2+2a^2r=0;$$

$$\therefore\ r=\pm1.$$

PLATE VIII.

(2) $$xy(x^2-y^2)+a(x-c)(x^2+y^2-a^2)=0.$$

In this curve the compartments are defined by the lines $x=0$, $y=0$, $x\pm y=0$, $x-c=0$, and $x^2+y^2-a^2=0$, all the points of intersection of the first three with the last two are points in the curve.

The asymptotes are $y+a=0$, which meets the curve where $x=\frac{a^2}{c}$; $x=0$, the next approximation to which is $x=-\frac{ac}{y}$; $x-y+a=0$, and $x+y-a=0$.

The only difficulty which arises with these data in tracing the curve is settled by fixing on which side of the asymptote $x+y-a=0$ the curve lies, and the next approximation giving $x+y=a-\frac{(a-2c)a}{2y}$, the curve is below the asymptote in $x'Oy$, when $c<\frac{1}{2}a$; and above, when $c>\frac{1}{2}a$.

When $c=\frac{1}{2}a$, since the three points $(a, 0)$, $(\frac{1}{2}a, \frac{1}{2}a)$, $(0, a)$ all lie in the line $x+y=a$, taking into account the two points at an infinite distance, this straight line would meet in one more point than the degree of the curve, which shews that the curve, for this value of c, contains the complete line $x+y-a=0$, the other part of the curve having for its equation

$$xy(x-y)+ax^2+\tfrac{1}{2}a^2(x-y)-\tfrac{1}{2}a^3=0,$$

$$\text{or}\quad (y+a)(x^2-\tfrac{1}{2}a^2)-x(y^2-\tfrac{1}{2}a^2)=0.$$

Fig. 3. The figure represents three forms of the curve, α denoting the case in which the curve divides into two, drawn with stronger lines, and β, γ denoting cases in which $c <$ and $> \frac{1}{2}a$.

Fig. 4. The next figure is for $c > a$.

(3) $$x^6-y^6+a^3(x^3+y^3-3axy)=0.$$

The three lines $x^3+y^3-3axy=0$, and the asymptotes $x\pm y=0$, make separate compartments, which are alternately occupied by the curve.

$$\text{Near } (-a, 0),\quad y=\xi,$$

$$\ldots\ldots\ (0, a),\quad x+\eta=0.$$


γ
b
β
a
γ

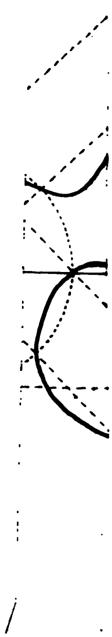


PLATE IX.

The dividing lines are dotted.

Note. The equation might be arranged

$$(x^3+y^3)(x^3-y^3+a^3)-3a^4xy=0,$$

and the form would be obtained perhaps more easily, if the point $x=y=\frac{2}{3}a$ were noticed. Fig. 1.

(4) $\quad x^6-y^6+a^2(x-a)(x^3+y^3-3axy)=0.$

$x-a$ is an additional line, for obtaining empty compartments, and since $x^3+a^2x-a^3=0$ has only one root which is positive, and near $(0,-a)$, $3\eta=4x$, the curve is easily drawn. Fig. 2.

(5) $(y-2x+2a)(x-2y+2a)(x^2+y^2)xy+b^3(x^3+y^3-a^3)=0.$

$$\text{Near } (0,a),\quad x^2a^2+b^3\eta=0;$$

this, with the symmetry with respect to $x=y$, is sufficient to determine the compartments within which the curve must lie, and therefore the sides of the asymptotes at which the curve comes into sight from an infinite distance.

The lines determining the compartments are dotted, except the axes. Fig. 3.

(6) $\quad (y^2-ax)^2(x-a)^2-a^2xy(x^2+y^2-a^2)=0.$

The circle and the axes determine the compartments, and two points coincide where any one of the three meet the curve.

The asymptotes are $(x-a)^2=0$, and $(y^2-ax)^2=0$,

$$\text{near } (a,\infty),\quad (x-a)^2=\frac{a^3}{y};$$

$$\ldots\ldots\ (\infty,\infty),\quad (ax-y^2)^2=a^2xy=ay^3,$$

$$ax=y^2\pm a^{\frac{1}{2}}y^{\frac{3}{2}},$$

$$\ldots\ldots\ (0,0),\quad x+y=0,\ \text{and}\ a^2x+y^3=0,$$

$$\ldots\ldots\ (a,0),\quad a^4\xi^2-(2a\xi+y^2)a^3y=0;$$

$$\therefore\ \xi-2y=0,\ \text{and}\ 2a\xi+y^2=0.$$

It may be seen by the compartments necessarily empty that the curve (when x is positive) is without the circle when y is positive, and within when negative. Fig. 4.

PLATE IX.

(7) $(x^2-y^2)(x-a)y^3-a^2(x^2+y^2-a^2)(x^2+y^2-4a^2)=0.$

Three of the dividing lines $x^2-y^2=0$, and $x-a=0$ are asymptotes, it is easily seen by the triangle, that $y^3-a^2x=0$ is a parabolic asymptote, or, still nearer, taking in the next term $-ax^2y^3$,

$$y^3=a^2(x+a).$$

If $x=0$, $y^5-a(y^2-a^2)(y^2-4a^2)$, the only root of which being the ordinate common to

$$y^2=ax \text{ and } (y-a)x^2+5a^2x-4a^3,$$

Fig. 5. is a little less than a.

It requires some care to determine in what direction certain elements of the curve should be joined; for instance, whether P, P' should be joined with Q, Q' and R, R' with S, S' or P, P' with R, R' and Q, Q' with S, S'.

For this purpose we might try $x=3a$, and shew that there are no negative values of y; or find where circles such as $x^2+y^2=\overline{3a}\,]^2$ or $(4a)^2$ meet the curve.

Note. It can be shewn that $x+2a=0$ meets the curve, where $3y^3-ay^2-12a^2y-3a^3=0$, and that this cubic has one positive root near $\frac{7}{3}a$, and two negative roots, one small, the other near $-\frac{3}{2}a$; $y+2a=0$ gives two large negative values of x, and two positive values a little greater than a.

ISOLATED PORTIONS.

173. In order to find whether there is an isolated closed portion of the curve, which does not cut either of the axes, the existence of which might be suspected on account of some compartments being empty which might have contained a portion of the curve, or for any other reason, several methods may be practically useful.

Since the curve is closed there must be two pairs of real points at which the curve is parallel to the axes, the points where it runs parallel to one of the axes will very often be able to be found, and the oval being once known to exist its nature can be examined by other processes.


x

PLATE IX.

174. Another plan is to draw a straight line from some point in the curve, a multiple point if there be one, and determine the directions when two of the other points of intersection become coincident, which would be feasible, *e.g.* for a curve of the fifth degree with a triple point; since the intersections of $y=mx$ with the curve would be given by a quadratic, if the origin were the triple point.

The directions so found would include those of all tangents drawn from the multiple point, and the curve being supposed traced with the exception of the oval, the two directions of the tangents to the oval could be selected.

Or, without seeking for the tangents, if any line parallel to one of the axes or to an asymptote, or drawn through a multiple point, could give, by assigning particular positions, two real points among the points of intersection, which do not coincide with any points previously determined, the oval would be detected.

175. In the following curve, the symmetry and the compartments which can be occupied shew where to look for an isolated oval:

$$(y-2x+2a)(x-2y+2a)x^2y^2+b(x^5+y^5+2a^3xy)=0.$$

The asymptotes are

$$y-2x+2a-\tfrac{11}{4}b=0,$$
$$x-2y+2a-\tfrac{11}{4}b=0,$$
$$2x^2=by, \text{ and } 2y^2=bx.$$

The next approximation to the form, $2a^3x+y^4=0$, at the origin, shews that the curve lies nearer to the axis of y than the dividing curve, when y is negative, and further when y is positive.

The compartments in which the roots can lie are cut by $y=x$; if then $y=x$,

$$(x-2a)^2x^2+2bx^3+2a^3b=0,$$

and the values of x are the abscissæ of the points of intersection of

$$(x-2a)^2+2bx=by \text{......................i,}$$
$$\text{and } x^2y+2a^3=0 \text{......................ii.}$$

PLATE X.

(ii) is a fixed curve, if a be fixed; (i) is a parabola which changes its shape and position with the value of b, the vertex

Fig. 1. being $(2a-b,\ 4a-b)$, and the latus rectum b. The figure is drawn for three positions of the parabola, marked α, β, γ. The case α is that of a multiple point.

The curve corresponding to β is drawn completely, those corresponding to α and γ are drawn only in the quadrant

Fig. 2. $x'Oy'$.

176. The following curve affords an example of an outlying oval, whose existence might not have been suspected:

$$xy^3 - 4x^3y + ay^3 + 3a^2xy + a^3x = 0.$$

The asymptotes are $y=0$, $x+a=0$, and $y \mp 2x \pm a = 0$.

Near $(0,\ 0)$, $y^3 + a^2x = 0$,

...... $(-a,\ a)$, writing the equation

$$(x+a)\,y^3 - 4xy\,(x^2-a^2) - a^2x\,(y-a) = 0,$$

the form is $\quad \xi - 4.2\xi + \eta = 0$, or $\eta = 7\xi$.

The curve cuts the asymptotes where $x=0$, $y=a$, $-\frac{1}{3}a$, $-a$, respectively, there being three points in each at an infinite distance.

The equations of the asymptotes lead us to put the equation of the curve in the form

Fig. 3.

$$(y-2x+a)\,(y+2x-a)\,(x+a)\,y + a^3\,(y+x) = 0.$$

(*a*) The curve is parallel to Oy where

$$4x\,(\tfrac{4}{3}x^2 - a^2)^3 = a^6\,(x+a),$$

whose real solutions are the common abscissæ of

$$x^2 = \tfrac{3}{4}a\,(y+a), \text{ and } 4xy^3 = a^3\,(x+a),$$

which gives roughly $x = -\frac{3}{5}a$, $-\frac{1}{5}a$, and $\frac{10}{9}a$, the breadth of the oval given by this is nearly $\frac{2}{5}a$.

(*b*) Or, the curve is parallel to Ox at the intersection of the two curves

$$y^3 + 3a^2y - 12x^2y + a^3 = 0,\quad 8x^3 + ay^2 = 0.$$

PLATE X.

The five points at which the curve is parallel to Ox are indicated by a, b, c, d, e, of which a, b belong to the oval.

(c) Or trying $x = \frac{1}{2}a$, $y^3 - 2a^2y - a^3 = 0$;

$$\therefore\ y = -a, \text{ and } \tfrac{1}{2}(\pm\sqrt{5}+1)\,a.$$ Fig. 4.

SPECIAL CURVE OF THE FOURTH DEGREE.

177. I shall conclude this chapter by shewing how the methods which have been explained work when applied to some equations selected by A. Beer, in a paper* on symmetrical curves of the fourth order, and serving excellently to illustrate the varieties which the general equation may produce. It will be found that all the peculiarities of the curves can be determined without any great difficulty.

The equations are

$$(y^2 \mp x^2)\{(\tfrac{1}{2}y-1)^2 \mp x^2\} + \mu(y+a) = 0.$$

The ambiguous signs allowing the equation to represent a curve with four, two, or no asymptotes.

178. $(y^2 - x^2)\{(\tfrac{1}{2}y-1)^2 - x^2\} + \mu(y+a) = 0.$

The curve being symmetrical with respect to Oy, we need only examine on which side of two of the asymptotes the curve lies, this is given by

$$y - x = \frac{2\mu}{3x^2}, \text{ and } \tfrac{1}{2}y - x - 1 = -\frac{\mu}{3x^2}.$$

The compartments, formed by the four asymptotes and the line $y + a = 0$, within which the curve can lie, are thus determined for the cases of μ positive and negative.

Where the curve meets Ox,

$$x^2 = \tfrac{1}{2}\{1 \pm \sqrt{(1-4\mu a)}\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{i},$$

so that, if $4\mu a$ be positive and not greater than 1, Ox is cut in four points; if μa be negative, it is cut in two points, one on each side of Oy.

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* Bonn, 1852.

PLATE X. the quadrants $x'Oy'$ and xOy, giving two points at which the curve is parallel to Ox, one in ED, the other in CF.

Fig. 5. If a smaller value of μ than this be chosen, (iii) does not meet (ii) and there is no point on Oy, but (v) meets (iv) in two points, giving points on CED and CF nearer to C than the former.

If a larger value of μ be taken, (ii) and (iii) intersect in two points, giving points on Oy', one on each side of the multiple point (iv) and (v) further from C than in the first curve.

If μ be negative, the critical case is when (ii) and (iii) touch, which they do at a point not far from the vertex of (ii), giving a conjugate point on Oy, the other points of intersection give a point on Oy' near O and a point at some distance on Oy beyond B. The reversed parabola (v) gives two points on CED and DG near D.

If a less value of μ be chosen, (ii) and (iii) give two values on OB, one on each side of the conjugate point; (iv) and (v) give two points on CED and DG nearer to D.

If a greater value of μ be taken, (ii) and (iii) give only two points on Oy' and Oy further from O than in the last case; (iv) and (v) give two points on CED and DG further from D.

Fig. 6. The figure represents by a darker line the critical cases, denoted by α and α', for μ positive and negative respectively, and by lighter lines the other cases in order denoted by β, γ and β', γ'; β and β' being the curves which approach towards the asymptotes.

The dotted line is the hyperbola (vi), on which lie all the points of contact of tangents to the different curves of the system which are parallel to Ox.

Fig. 5. 181. It can be seen, from the figure of the fixed curves, that when a is negative, a parabola (iii) can be drawn with some negative value, if a be small, which will touch the parabola (ii), and cut it in two other points, and that, for a particular value of a, the particular parabola which touches

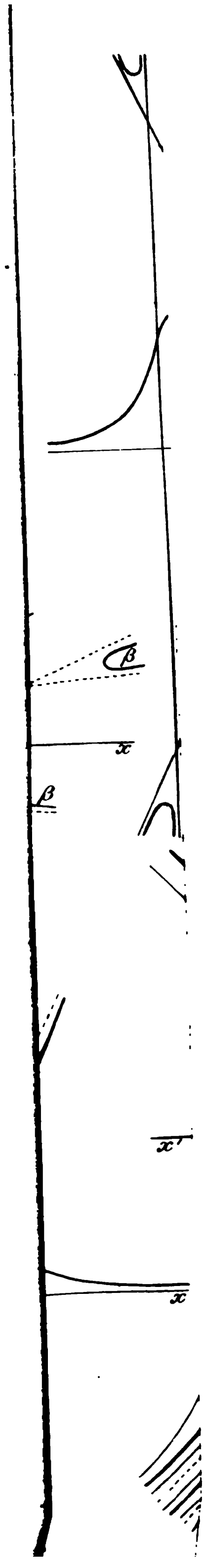

β
x
β
x'
x

PLATE X.

(ii) will meet it in three consecutive points, the corresponding point of the curve being a cusp on Oy.

Fig. 7. The figure represents the principal part of the forms for μ which corresponds to the cusp, denoted by α, and for two other values, one small, in which the curve β approaches the asymptotes, and one large denoted by γ.

182. There will also be a cusp for some negative value of a, a little less than 2, and μ positive; and for this value of a, some value of μ will give contact between the parabolas (iv) and (v), and therefore a multiple point.

PLATE XI.

Fig. 1. In the figure the curves marked α and ω are the cases of cusp and multiple point respectively, β denotes the curve which approaches the asymptotes, γ that which corresponds to a large value of μ, making (iv) and (v) intersect in two points on each side of Oy, and therefore pointing out four points at which the curve is parallel to Ox, situated on the dotted line which is the hyperbola (vi).

A fourth curve is denoted by δ, being one which bends towards the cusp α and the multiple point ω.

183. The directions of the curve at the principal points are as follows, for the purpose of accurate drawing.

Near the point $(\alpha, 0)$ where the curve cuts Ox, since the equation may be written

$$x^4 - x^2 + \mu a + (x^2 + \mu)\, y + \text{terms in } y^2 = 0,$$

$$\text{or } (x^2 - \alpha^2)(x^2 + \alpha^2 - 1) + (x^2 + \mu)\, y + \ldots = 0;$$

$$\therefore\ (2\alpha^2 - 1)\, 2\alpha x + (\alpha^2 + \mu)\, y = 0,$$

$$\text{or } \pm\sqrt{(1 - 4\mu a)}\, 2\alpha x + (\alpha^2 + \mu)\, y = 0 \text{ gives the tangent.}$$

Near $(a, -a)$, let $x = a + \xi$, $y = -a + \eta$,

$$2a(\xi + \eta)\{a^2 - (\tfrac{1}{2}a + 1)^2\} + \mu\eta = 0.$$

Near $(-a, -a)$, if $x = -a + \xi$, $y = -a + \eta$,

$$-2a(\xi - \eta)\{a^2 - (\tfrac{1}{2}a + 1)^2\} + \mu\eta = 0.$$

Near $(\tfrac{1}{2}a + 1, -a)$, $-\{a^2 - (\tfrac{1}{2}a + 1)^2\}(a + 2)(\tfrac{1}{2}\eta + \xi) + \mu\eta = 0.$

PLATE XI. Near $\{-(\frac{1}{2}a+1),\ -a\}$, $\ -\{a^2-(\frac{1}{2}a+1)^2\}(a+2)(\frac{1}{2}\eta-\xi)+\mu\eta=0.$

Near $(0, \beta)$, where $\beta^2(\frac{1}{2}\beta-1)^2+\mu(\beta+a)=0$,

$$\{\beta^2+(\frac{1}{2}\beta-1)^2\}x^2-\{\beta(\beta-1)(\beta-2)+\mu\}\eta=0,$$

for an ordinary point;

$$\{\beta^2+(\frac{1}{2}\beta-1)^2\}x^2-\{3(\beta-1)^2-1\}\eta^2=0,$$

for a multiple or conjugate point, according as

$$(\beta-1)^2 > \text{ or } < \tfrac{1}{3};$$

$$\{\beta^2+(\frac{1}{2}\beta-1)^2\}x^2-6(\beta-1)\eta^3=0,$$

for a cusp, upwards or downwards, as $\beta >$ or < 1.

The values of a, μ, and β for a cusp on Oy may be determined at once by the three equations

$$3(\beta-1)^2=1,\quad (\beta-1)^3-(\beta-1)+\mu=0,$$

$$\text{and}\ \{(\beta-1)^2-1\}^2+4\mu(\beta-1+a+1)=0;$$

$$\therefore\ \beta-1=\pm\tfrac{1}{3}\sqrt{3}=\pm\tfrac{4}{7}\ \text{nearly},$$

$$\mu=\pm\tfrac{2}{9}\sqrt{3}=\pm\tfrac{8}{21}\ \ldots\ldots,$$

$$a=-1\mp\tfrac{1}{2}\sqrt{3}=-(2-\tfrac{1}{8})\ \text{or}\ -\tfrac{1}{8}\ \text{nearly}.$$

184. The algebraical determination of the singular points is as follows:

The equations to be satisfied besides that of the curve are

$$2y\{(\tfrac{1}{2}y-1)^2-x^2\}+(\tfrac{1}{2}y-1)(y^2-x^2)+\mu=0,$$

$$\text{or}\ y(y-1)(y-2)-(\tfrac{5}{2}y-1)x^2+\mu=0,$$

$$\text{and}\ x\{2x^2-y^2-(\tfrac{1}{2}y-1)^2\}=0;$$

and μ is to be determined so that these equations and the equations of the curve are simultaneously satisfied.

$$(a)\qquad x=0,\ y(y-1)(y-2)+\mu=0,$$

$$\text{and}\ y^2(\tfrac{1}{2}y-1)^2+\mu(y+a)=0;$$

$$\therefore\ y(y-2)-4(y-1)(y+a)=0,$$

so that when a is given, the last equation gives the values of y, and the first gives the corresponding values of μ. The last equation gives

$$y=\tfrac{1}{3}[-2a+1\pm\sqrt{\{4(a+1)^2-3\}}],$$

$$\text{and}\ 4(a+1)^2=3\ \text{for a cusp}.$$

PLATE XI.

(*b*) $$2x^2 - y^2 - (\tfrac{1}{2}y - 1)^2 = 0;$$

$$\therefore\ y^2 - x^2 = -\{(\tfrac{1}{2}y - 1)^2 - x^2\} = \tfrac{1}{2}\{y^2 - (\tfrac{1}{2}y - 1)^2\};$$

$$\therefore\ \tfrac{1}{2}(\tfrac{3}{2}y + 1)(\tfrac{3}{4}y^2 + y - 1) = \mu;$$

and from the equation of the curve,

$$\tfrac{1}{4}(\tfrac{3}{4}y^2 + y - 1)^2 = \mu(y + a);$$

$$\therefore\ \tfrac{3}{4}y^2 + y - 1 = (3y + 2)(y + a),$$

$$\text{and } y = \tfrac{2}{9}[-(3a + 1) \pm \surd\{(3a - 2)^2 - 12\}],$$

which gives the position of the singular point for any value of a.

In the particular case of the cusp

$$3a - 2 = \pm 2\surd 3,\ a = \tfrac{11}{6},\ \text{or } -\tfrac{1}{2};$$

$$\therefore\ y = \tfrac{2}{9}(-3 \mp 2\surd 3) = -\tfrac{2}{3} \mp \tfrac{7}{9} \text{ nearly},$$

$$3y + 2 = \mp \tfrac{4}{3}\surd 3,$$

$$\therefore\ 9y^2 + 12y + 4 = \tfrac{16}{3},\ \text{or } \tfrac{3}{4}y^2 + y - 1 = \tfrac{11}{9} - 1;$$

$$\therefore\ \mu = \pm \tfrac{8}{27}\surd 3 = \pm \tfrac{14}{27} \text{ nearly}.$$

185. The figures represent the systems of curves which can be drawn for different values of μ, both positive and negative, in the two cases in which a cusp occurs which is not in the axis Oy, viz. when $a = \frac{2}{3}(\pm\surd 3 + 1)$. Figs. 2, 3.

The multiple point in Oy' belongs to the positive value of μ, the conjugate point in Oy to the negative system.

186. The only other case which need be considered is the case of $a = \infty$. Here μ is indefinitely small and μa finite if we wish to bring the curve within sight.

The equation thus becomes

$$(y^2 - x^2)\{(\tfrac{1}{2}y - 1)^2 - x^2\} = \pm c^2 \text{ .... (a)},$$

$$y = 0,\ x^2 = \tfrac{1}{2}\{1 \pm \surd(1 \pm 4c^2)\},$$

$$x = 0,\ y = 1 \pm \surd(1 \pm 2c),$$

with the upper sign of (a).

Two values of x^2 are equal each to $\frac{1}{2}\{y^2 + (\frac{1}{2}y - 1)^2\}$, if $y^2 - (\frac{1}{2}y - 1)^2 = \pm 2c$, with the lower sign of (a),

$$\text{or } y = \tfrac{2}{3}\{-1 \pm \surd(4 \pm 6c)\}.$$

PLATE XI. Fig. 4.

The figure is drawn for three values of c, both for the upper and lower sign; the dark lines and conjugate point α correspond to the upper, with a value $c=\frac{1}{2}$, the dark lines and conjugate point β to the lower, with a value $c=\frac{2}{3}$.

187. The case of two asymptotes is given by the equation

$$(y^2+x^2)\{(\tfrac{1}{2}y-1)^2-x^2\}+\mu(y+a)=0.$$

The side of the asymptotes on which the curve lies is given by

$$\tfrac{1}{2}y+x-1=\frac{4\mu}{5x^2};$$

where the curve meets Ox,

$$x^2=\tfrac{1}{2}\{1\pm\sqrt{(1+4\mu a)}\} \text{ .... i.}$$

The curve meets Ox in two points if $4\mu a$ be positive, in four points if it be negative and >-1, and in no point if it be <-1.

Where it meets Oy,

$$y^2(\tfrac{1}{2}y-1)^2+\mu(y+a)=0,$$

the ordinates, as before, are the common ordinates of

$$(y-1)^2=x+1 \text{ .... ii,}$$

$$\text{and } x^2+4\mu(y+a)=0 \text{ .... iii.}$$

Writing the equation in the form

$$[x^2+\tfrac{1}{2}\{y^2-(\tfrac{1}{2}y-1)^2\}]^2-\tfrac{1}{4}\{y^2+(\tfrac{1}{2}y-1)^2\}^2-\mu(y+a),$$

we see that x^2 has equal values, or the curve is parallel to Ox, or each has a singular point, where

$$\{y^2+(\tfrac{1}{2}y-1)^2\}^2+4\mu(y+a)=0;$$

therefore the ordinates of such points are the common ordinates of

$$y^2+(\tfrac{1}{2}y-1)^2=x,$$

$$\text{or } (y-\tfrac{2}{5})^2=\tfrac{4}{5}(x-\tfrac{4}{5}) \text{ .... iv,}$$

$$\text{and } x^2+4\mu(y+a)=0 \text{ .... v,}$$

the points themselves are on the ellipse

$$2x^2+\tfrac{3}{4}y^2+y-1=0,$$

$$\text{or } \frac{x^2}{\frac{2}{3}}+\frac{(y+\frac{2}{3})^2}{(\frac{4}{3})^2}=1 \text{ .... vi.}$$

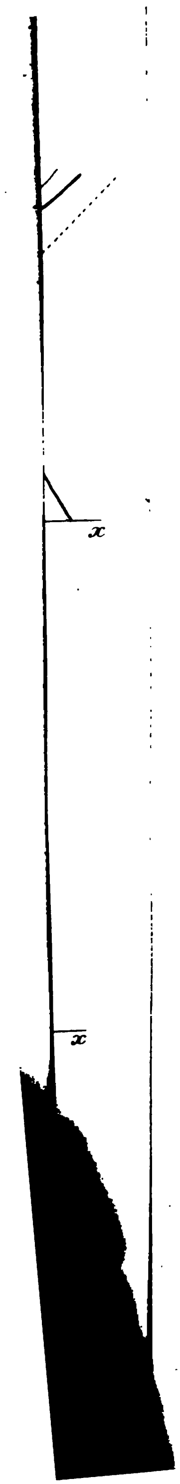

x
x

PLATE XII.

188. The fixed parabolas (ii) and (iv) and the ellipse (vi) are represented in the figure, the parabolas touch one another at the point $(8, -2)$. The parabola which varies its magnitude with μ is in the same direction for the determination both of the points of intersection with Ox and of the points where the curve is parallel to Ox. Fig. 1.

It will be sufficient to shew in three cases, viz. when a is 1, -3, and ∞, how to form the systems of curves corresponding to different values of μ.

189. When $a = 1$, the variable parabola has four critical magnitudes, when it touches (iv) in some points P and U, and (ii) in some points S and T; lines parallel to Ox through these points determine p, u, on the ellipse (vi), which are multiple points, and s, t, on the axis of y which are conjugate points. The curves corresponding to values of μ which make the parabolas (iv) and (v) touch, are drawn with a dark line, the conjugate points are marked s and t. Fig. 1. Fig. 2.

The curves which belong to other values of μ bend towards the asymptotes or the dark curves, the points where the curves cross the dotted line spu are the points of contact of tangents parallel to Ox.

190. When $a = -3$, the critical magnitudes, μ being positive, are where the parabolas touch at some points Q and R, determining the multiple point q and conjugate point r, there is also a multiple point on Oy corresponding to contact with the upper branch of (ii), the dotted line rq is the ellipse which contains the points at which the curves are parallel to Ox. Fig. 1. Fig. 3.

191. The case of $a = \infty$ may be found from the equation

$$(y^2 + x^2)\{(\tfrac{1}{2}y - 1)^2 - x^2\} = \pm c^2 \quad \text{............ (a)},$$

$$y = 0, \quad x^2 = \tfrac{1}{2}\{1 \pm \sqrt{(1 \mp 4c^2)}\},$$

$$x = 0, \quad y = 1 \pm \sqrt{(1 \pm 2c)},$$

with the upper sign of (a).

PLATE XII.

Two values of x^2 are equal each to $\frac{1}{2}\{(y-1)^2 - y^2\}$ with the upper sign of (a), if

$$y^2 + (\tfrac{1}{2}y - 1)^2 = 2c, \text{ or } y = \tfrac{2}{5}\{1 \pm \sqrt{(10c - 4)}\}.$$

So that when the upper sign is taken, the curve does not meet Ox unless $2c =$ or < 1, if $2c = 1$, there is a conjugate point at $(0, 1)$.

There is a multiple point when $5c = 2$ at a point near $(\frac{1}{2}, \frac{2}{5})$.

The curve is of the form $x^4 = -\frac{3}{2}\eta$, when $c = 4$ and $y = -2$, which gives the critical case where the curve ceases to bend upwards, so as to lose a point where it is parallel to Ox.

When the lower sign is taken there is no singular point.

Fig. 4. The figure represents the principal varieties, the critical cases of $c = \frac{1}{2}$ and $\frac{2}{5}$ being marked α and β.

192. The remaining case of no asymptote is given by

$$(y^2 + x^2)\{(\tfrac{1}{2}y - 1)^2 + x^2\} + \mu(y + a) = 0.$$

The points of intersection with Ox are given by

$$x^2 = \tfrac{1}{2}\{-1 \pm \sqrt{(1 - 4\mu a)}\} \text{ .... i.}$$

The values of x^2 are only possible for values of y which make $y^2(y - 2)^2 + 4\mu(y + a)$ negative, $= -\alpha^2$ suppose, and if β be given by $\alpha^2 + 4\mu(\beta + a) = 0$, these values of y will satisfy $y^2(y-2)^2 + 4\mu(y - \beta) = 0$, and be the common ordinates of the parabolas.

$$x^2 + 4\mu(y - \beta) = 0 \text{ .... ii,}$$

$$\text{and } (y - 1)^2 = x + 1 \text{ .... iii,}$$

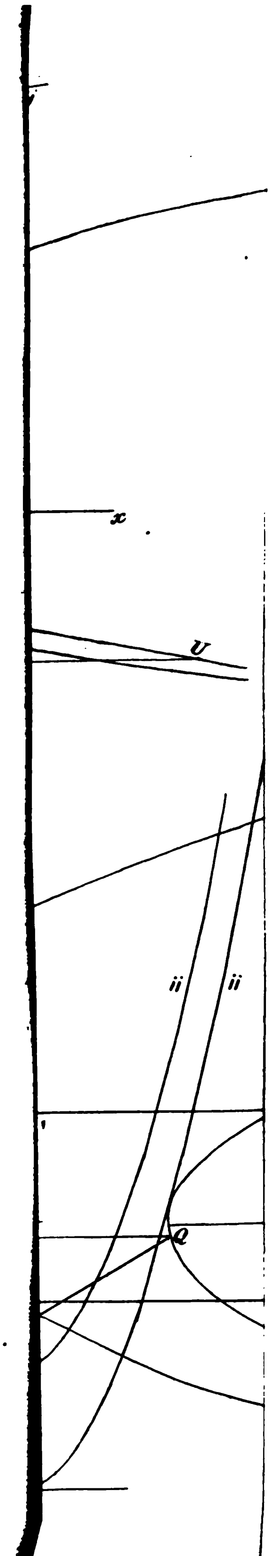
and with these values

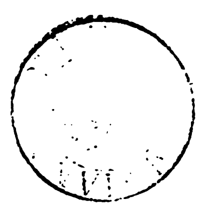
$$x^2 = \tfrac{1}{2}[\sqrt{\{\alpha^2 + (y^2 + (\tfrac{1}{2}y - 1)^2\}^2}] - \{y^2 + (\tfrac{1}{2}y - 1)^2\}]$$

$$= \tfrac{1}{2}\{\sqrt{(\alpha^2 + \xi^2)} - \xi\} \text{ .... iv,}$$

$$\text{if } (y - \tfrac{2}{5})^2 = \tfrac{4}{5}(\xi - \tfrac{4}{5}) \text{ .... v.}$$

193. The parabola (iii) is fixed for all values of a and μ; the parabola (ii) for particular values of a and μ commences in the position where its vertex is at $(0, -a)$, and


x
U
ii
ii
Q



moving in the direction of its axis, gives, by its intersections with (iii), successive values of y which give real values of x; the initial position giving the points where the curve meets the axis of y. PLATE XII.

No values of y give equal values of x differing from zero, so that the curve is never parallel to Ox, except where it meets the axis of y.

The equation giving the points where the tangent to the curve is parallel to Oy is of too high an order to solve approximately except in particular cases.

194. The equations (iv) and (v) give means of determining by construction the value of x corresponding to any value of y.

Construct the parabolas (iii), (v), and (ii), the last for two positions, viz. when $\beta = -a$, and when it has a general value. Fig. 5.

If P be one of the points of intersection of (ii) and (iii), let PMQ parallel to Ox meet (v) in Q and Oy in M, and let βR tangent at β, the vertex of (ii), intersect the parabola (ii) in its first position, viz. aR in R, take S in aM such that $MS = \beta R$, and T in PR, so that $QT = QS$.

Then $x^2 = \frac{1}{2}TM$ corresponds to $y = OM$; and x is the ordinate UN to the abscissa $AN = \frac{1}{2}TM$ in the parabola (iii).

195. To shew how to trace the curve we will take the case in which the parabola (ii) in its first position, when $\beta = -a$, touches (iii) as at A, μ being negative, and cuts it in B, C; Aa, Bb, Cc determine points where the curve cuts the axis of y, of which a is a multiple point, as (ii) moves upwards it cuts (iii) in P, Q, R, S; as (ii) continues to move upwards, P and Q move away from A in opposite directions, and R and S from B and C towards O and D; when (ii) touches (iii), R and P meet as at T, the figure traced out by the points which correspond to them being *aprb* and the symmetrical half loop; similarly, Q and S meet when (ii) touches (iii) with the last contact as at U, and they produce the top loop *asc*. PLATE XIII. Fig. 1.

PLATE XIII.

It may be seen, from the last article, that the maximum value of x in the top loop is very nearly opposite to the point of last contact, on account of the large value of ξ.

196. The system of curves which have μ for their parameter, when a is a given positive quantity, are readily traced, a guide to the general direction being the fact that, as $-\mu$ increases from O to the value which corresponds to the figure of eight, called a lemniscate, which has just been formed, the curve changes from two conjugate points at O and D to the lemniscate.

When μ is negative and small, the points in which the initial position of (ii) meets (iii) are two pairs of points, one on each side of and near both O and B, giving rise to two small ovals surrounding the conjugate points.

As $-\mu$ approaches the value for the lemniscate, the ovals terminate near c, a, and b.

When $-\mu$ is greater than this value, the initial position of (ii) meets (iii) in only two points R' and S', joining points on Oy below b and above c. As (ii) moves upwards, R' moves towards O until (ii) and (iii) touch, which they may do if μ be not too great, the two points P, Q then come into existence at first coincident, and then P meeting R, Q meeting S as before.

Fig. 2. The figure represents two of the ovals, the lemniscate, and two curves beyond.

The case of μ positive is the conjugate point and oval marked + in the figure.

197. The case of a cusp on the axis of y occurs when $a = -1 \pm \frac{1}{2}\sqrt{3}$ or $-\frac{1}{8}$, and $-\frac{15}{8}$ nearly, as shewn in Art. 184.

Fig. 3. The figure is drawn for the case of $a = -1 + \frac{1}{2}\sqrt{3}$, shewing the systems for negative and positive values of μ.

The figures are somewhat similar for $a = -1 - \frac{1}{2}\sqrt{3}$, differing principally in the breadth, and in the position where it is greatest.

PLATE XIII.

198. When a is between $-1 \pm \frac{1}{2}\sqrt{3}$, the systems for μ positive and negative are each a conjugate point and a series of ovals. Fig. 4.

199. The case of $a = \infty$ is given by

$$(y^2 + x^2)\left(\tfrac{1}{4}y - 1)^2 + x^2\right) = c^2,$$

$$y = 0, \quad x^2 = \tfrac{1}{2}\{-1 + \sqrt{(1 + 4c^2)}\} = \alpha^2, \text{ suppose},$$

$$x = 0, \quad y^2 - 2y = \pm 2c, \quad y = 1 \pm \sqrt{(1 \pm 2c)}.$$

$$\text{Near } (\alpha, 0), \quad -\alpha^2 y + (4\alpha^3 + 2\alpha)\,\xi = 0,$$

$$\ldots\ldots\ (0, 1), \text{ when } 2c = 1,$$

$$\tfrac{5}{4}x^2 - \eta^2 = 0, \text{ or } \eta = \tfrac{3}{2}x \text{ nearly.}$$

In this case we can construct for the points where the curve is parallel to Oy by means of the equations

$$4(5y - 2)^2 c^2 = y(2 - y)(3y^2 + 4y - 4)^2 \ldots\ldots\ (1),$$

$$\text{and } (5y - 2)x^2 + 2y(y - 1)(y - 2) = 0 \ldots\ldots\ (2),$$

which are easily obtained from the condition.

The solutions of equation (1) are the common ordinates of the hyperbola

$$x(5y - 2) = 3y^2 + 4y - 4,$$

$$\text{or } (5y - 2)\left(x - \tfrac{3}{5}y - \tfrac{26}{25}\right) = -\tfrac{48}{25} \ldots\ldots\ldots\ (3),$$

and the curve $\qquad x^2 y(2 - y) = 4c^2 \ldots\ldots\ldots\ldots\ldots\ (4).$

Lines drawn parallel to Ox through the points of intersection of (3) and (4) meet the curve (2) in points where the tangents to the given curve are parallel to Oy.

The figure contains portions of the branches of the hyperbola (3), the curve (2), viz. Oqu and $astp$, and the curve (4) traced for $c = \frac{1}{2}$ and $\frac{3}{8}$, viz. PAQ and $P'Q'$. Fig. 5.

When $2c = 1$ the curve has a lemniscate form, and the points where it is parallel to Oy are p, q corresponding to the intersections P, Q; the point a corresponds to A, and, being the node of the lemniscate, the condition of equal values of y is satisfied.

When $c = \frac{3}{8}$, the only points of intersection which give maximum values of x are P', Q', the points on the curve being p', q'.

PLATE XIII.

When $c > \frac{1}{2}$, the curves (3) and (4) first intersect in R and S and the lower branch, and, as c increases, they touch in T, and afterwards have only one value in the branch QU. The corresponding points in the given curve are r, s, t, u.

Fig. 6. The complete system is drawn with the same letters r, s, ...

Note. The curve (4) is easily drawn for different values of c, since, by transposing the origin to (0, 1), the equation is $4c^2 = (1 - y^2)\,x^2$, if then $y = \sin\theta$, $x = 2c\sec\theta$.

X.

(1) Find the multiple points of the curve

$$(x^2 - a^2)^2\,y = (y^2 - b^2)^2\,x,$$

and trace the curve.

(2) Find the triple point in the curve

$$y^4 + x^4 - 4ay^3 + 2ay^2x + 2ax^3 + 8a^2y^2 - 4a^2xy - 8a^3y + 2a^4 = 0,$$

and trace the curve.

(3) Find the point of inflexion of the curve

$$ay\,(x - a) = x^2\,(x + a).$$

(4) Apply the method by compartments to draw the curves

(*a*) $$xy^3 = c^2\,(x^2 + y^2 - a^2).$$

Find the value of c in order that there may be a singular point.

(*b*) $$x^4 - y^4 - ax\,(x^2 + y^2 - c^2) = 0.$$

(*c*) $$x^4 - y^4 - xy\,(x^2 + y^2 - c^2) = 0.$$

(*d*) $$2\,\{x^2 - (y-a)^2\}\,\{4x^2 - (y-2a)^2\} + 9y^2\,\{x^2 + a\,(y-a)\} = 0.$$

Find the multiple points, and the points where the curve is parallel to Ox.

(5) Draw the systems of curves corresponding to different values of μ in the curves

$$(x^2 \pm y^2)\,\{x^2 \pm (\tfrac{1}{2}y - 1)^2\} + \mu\,(y + 3) = 0,$$

$$\text{and}\;\; (x^2 \pm y^2)\,\{x^2 \pm (\tfrac{1}{2}y - 1)^2\} + \mu y = 0.$$


iii

CHAPTER XI.

SYSTEMATIC TRACING OF CURVES. REPEATING CURVES.

PLATE XIV.

200. In the preceding chapters, when giving examples of the particular points upon which I was engaged, I thought it would be more interesting to the student to see how the part under examination fitted into the rest of the curve considered, and I therefore have given throughout a number of elements of the curves, which have frequently been sufficient to determine the entire shape; at the same time, I have generally expected the student only to examine the figure, in order to see how the elements were combined. I shall now give a few rules for the systematic treatment of the equation of a curve, only recommending them as convenient plans to adopt, leaving to the ingenuity of the student such modifications as particular forms of equations may suggest.

201. The discussion of those equations which can be readily solved with respect to either of the coordinates, or in which the coordinates may be separated from each other, can be conveniently arranged in the following order:

i. The statement of any symmetry which may be observed in the equation of the kinds mentioned in Art. 6.

Any fresh arrangement of the terms of the equation which may from the form suggest properties of the curve, for instance, if it could be arranged in the form $u^2+vw=0$, in which case $v=0$ and $w=0$ would be curves which would touch the required curve at their points of intersection with $u=0$; or, if any arrangement would give distinct compartments within which the curve must lie.

PLATE XIV.

ii. The tabulation of particular points such as those which lie on the axes, or which the form of the solution of the equation may suggest.

iii. The form of the curve at all the points at a *finite* distance which may seem to be most important.

iv. The position of the asymptotes rectilinear or parabolic, when the fact of the curve going off to infinity in any direction has been established in the course of tabulation. If not obvious for other reasons, the side of the asymptote on which the curve lies must be investigated.

v. If more than the general form be looked for, it may be necessary to find such points as where the curve is parallel to the axes, and other peculiar points which would not be required in a rough determination of the locus.

202. The following curves, selected to illustrate special difficulties, will illustrate the application of these rules, commencing with the curve promised to be traced in Art. 49.

(1) $$y(b^2 - y^2) = x^2(a - x).$$

i. There is no symmetry, but the lines $y = 0$ or $\pm b$, and $x = a$ determine compartments within which the curve must lie.

ii. $x = 0$ or a, $y = 0$ or $\pm b$,

$x = \infty$, $y = \infty$.

iii. Near $(0, 0)$, $b^2y = ax^2$,

...... $(0, \pm b)$, $-2b^2y = ax^2$,

...... $(a, 0)$, $b^2y = -a^2\xi$,

...... $(a, \pm b)$, $2b^2\eta = a^2\xi$.

iv. ...... (∞, ∞), $y - x + \frac{1}{3}a = 0$.

The asymptote meets the curve at a finite distance, where

$$-\tfrac{1}{3}a(y^2 + xy + x^2) + ax^2 - b^2y = 0,$$

$$\text{or } a(y^2 + xy - 2x^2) + 3b^2y = 0,$$

$$\text{or } a^2(y + 2x) - 9b^2y = 0,$$

$$\text{or } (9b^2 - a^2)y = 2a^2x.$$

PLATE XIV.

The three points of intersection are at an infinite distance if this equation is $y = x$, or if $3b^2 = a^2$.

v. The curve is parallel to Ox, or there is a singular point, where $2ax - 3x^2 = 0$.

It is parallel to Oy, or there is a singular point, where $b^2 - 3y^2 = 0$.

The singular point occurs when $\left(\frac{2a}{3}, \frac{b}{\sqrt{3}}\right)$ is a point on the curve, in which case

$$2a^3 = 3\sqrt{3}b^3.$$

The figures are drawn for the cases of $b = a$, $b\sqrt{3} = a$, and $2a^3 = 3\sqrt{3}\,b^3$ denoted by β, γ and α respectively. Fig. 1.

(2) $$x^4y^3 = a^2(a-x)^3(a-2x)^2.$$

The compartments are determined by $x = a$ and $\frac{1}{2}a$, and $y = 0$.

$$x = 0, \quad y = \infty,$$
$$x = a, \quad y = 0,$$
$$x = \tfrac{1}{2}a, \quad y = 0,$$
$$x = \infty, \quad y = \infty.$$

Near $(a, 0)$, $y^3 = -\xi^3$,

...... $(\frac{1}{2}a, 0)$, $y^3 = 8a\xi^2$,

...... $(0, \infty)$, $x^4 = \dfrac{a^7}{y^3}$,

...... (∞, ∞), $y^3 = -a^2x$.

The curve is parallel to Ox where Fig. 2.

$$2x^2 + 5ax - 4a^2 = 0.$$

(3) $$(x^2 - a^2)^2 + (y^2 - b^2)^2 = a^4.$$

The equation may be written

$$(y^2 - b^2)^2 = (2a^2 - x^2)\,x^2.$$

The curve is symmetrical with respect to both axes,

$$x = 0 \text{ or } a\sqrt{2}, \quad y = b \text{ two values},$$
$$y = 0, \quad x^2 = a^2 \pm \sqrt{(a^4 - b^4)},$$
$$y^2 = b^2 \pm a^2, \quad x = a \text{ two values}.$$

PLATE XIV.

Near $(0, b)$, $4b^2\eta^2 = 2a^2 x^2$,

...... $(a\sqrt{2}, b)$, $4b^2\eta^2 = 2a^2(-2\sqrt{2}\, a\xi)$.

Fig. 3. The figures are drawn for the cases $a = b$, $b\sqrt{2}$, and $\dfrac{b}{\sqrt{2}}$, denoted by α, β, and γ.

(4) $$a^2y^2(x-b)^2 = (a^2 - x^2)(bx - a^2)^2.$$

The curve is symmetrical with respect to the axis of x,

$$b > a,\quad x = 0,\quad y = \frac{a^2}{b},$$

$$x = \frac{a^2}{b},\quad y = 0,$$

$$x = a,\quad y = 0,$$

$$x = 0 \text{ to } a,\quad y \text{ is real},$$

$$b < a,\quad x = 0,\quad y = \frac{a^2}{b},$$

$$x = b,\quad y = \infty,$$

$$x = a,\quad y = 0,$$

$x > a$, y is impossible unless $x = \dfrac{a^2}{b}$, where there is a conjugate point.

When $b = a$, the curve divides into two coincident straight lines, and a circle.

Near $\left(0, \dfrac{a^2}{b}\right)$, $y = \dfrac{a^2}{b} - \left(1 - \dfrac{a^2}{b^2}\right)x$,

...... $\left(\dfrac{a^2}{b}, 0\right)$, $\left(1 - \dfrac{a^2}{b^2}\right)y^2 = \xi^2$,

...... $(\pm a, 0)$, $y^2 = \mp 2a\xi$,

...... (b, ∞), $x - b = \dfrac{(a^2 - b^2)^{\frac{3}{2}}}{ay}$.

Fig. 4. The dark lines correspond to $b = a$, the curves marked α and β correspond to $b >$ and $< a$.

PLATE XIV.

(5) $$(a^3y-x^4)^2-a^4(x-2a)^2(x^2-a^2)=0.$$

No value of x is between a and $-a$,

$$x=\pm a,\quad y=a \quad \text{two values,}$$
$$x=2a,\quad y=2^4a \;\ldots\ldots\ldots\ldots,$$
$$x=\infty,\quad y=\infty,$$
$$y=0 \text{ gives no real values of } x.$$

Near $(\pm a, a)$, $\eta^2=\pm 2a\xi$,

...... $(2a, 2^4a)$, $(a^3\eta-4.2^3a^2\xi)^2-3a^6\xi^2=0$,

or $\eta=(2^5\pm\sqrt{3})\,\xi$.

Near (∞, ∞), $a^3y-x^4=\pm a^2x^2$.

There is, therefore, a multiple point on the parabolic asymptote $a^3y=x^4$, from which the curve runs off on both sides of the asymptote.

The figures marked α and β correspond to the cases of $b>$ and $<a$. The darker lines belong to the critical case $b=a$. Fig. 5.

(6) $$(x^2-4)^2-y^2(y+16)=0.$$

The curve is symmetrical with respect to Oy; therefore considering only positive values of x,

$$y=0 \text{ or } -16,\quad x=2 \text{ two values,}$$
$$x=0,\quad y^3+16y^2=16,$$
$$\therefore\ y=1-\tfrac{1}{35},\quad -(1+\tfrac{1}{29}),\quad -(16-\tfrac{1}{16}) \text{ nearly,}$$
$$y=\infty,\quad x=\infty.$$

Near $(2, 0)$, $(4\xi)^2-16y^2=0$,

...... $(2, -16)$, $(4\xi)^2-16^2\eta=0$.

If β be one of the values of y when $x=0$,

near $(0, \beta)$, $-8x^2-(3\beta^2+32\beta)\,\eta=0$,

$$\therefore\ 4x^2+17\eta=0,$$
$$4x^2-15\eta=0,$$
$$\text{and } x^2+32\eta=0 \text{ nearly.}$$

Near (∞, ∞), $x^4-y^3=0$.

PLATE XIV. Fig. 6.

It may be shewn that the curve is parallel to Oy, where $y = -\frac{31}{5}$ and $x = \frac{16}{5}$ nearly.

203. In the case of equations which cannot be solved with respect to either of the coordinates, or which, if capable of solution, would lead to clumsy results, no order of proceeding can be laid down as being generally the best; but it must be good, in the first place, to look for symmetry in an equation, and to examine whether the equation can be rearranged so as to exhibit properties, or whether a change of coordinate axes would simplify it; in the second place, to find whether the curve cuts the axes and where, and to discover whether the curve passes off to infinity, either directly, or, if there be many terms or any doubt exists, by means of the Analytical Triangle. It will generally be best next to find some of the most important forms of the curve near such particular points as may have turned up, and the side of the asymptotes on which the curve lies at each end, or, instead of this, the points where the curve cuts the asymptotes at a finite distance; it will then be seen whether it is necessary to go into the question of the points where the curve is parallel to the axes or whether there are any singular points.

204. The example given at the end of the last chapter is a good illustration of the methods of meeting difficulties, but a few more examples will not be useless, which have been selected with a view of meeting as many difficulties as possible in a short space.

(1) $$(y^2 - ax)^2 + (x^2 - ay)^2 = a^4.$$

The curve is symmetrical with respect to $x = y$,

$$x = 0, \quad y^2 = \tfrac{1}{2}(\sqrt{5} - 1)\, a^2 = \tfrac{5}{8}a^2 \text{ nearly},$$

$$\therefore\ y = \tfrac{9}{16} a \sqrt{2} \text{ nearly},$$

$$x = y = \tfrac{1}{2}\{1 \pm \sqrt{(2\sqrt{2} + 1)}\}\, a = \tfrac{3}{2}a \text{ or } -\tfrac{1}{2}a \text{ roughly}.$$

$$\text{If } x^2 - ay = \pm a^2, \quad (y^2 - ax)^2 = 0,$$

PLATE XIV.

the first parabola, therefore, touches the curve where

$$y^4 - a^3y \mp a^4 = 0.$$

The lower sign gives no solution; but with the upper sign

$$y = -\tfrac{3}{4}a \text{ and } \tfrac{6}{5}a \text{ nearly,}$$

$$x = \tfrac{9}{16}a \text{ and } \tfrac{3}{2}a \text{ nearly,}$$

and $x^2 = a(y + a)$ touches the curve.

Near $(0, a)$, $-2aa^2x + 4a^3\eta + 2a^2a\eta = 0$;

$$\therefore \eta = \frac{aa}{2a^2 + a^2}x = \frac{4a}{9a}x.$$

In the figure the parabola AB is drawn touching the curve at P and Q where $y^2 = ax$ meets it. Fig. 7.

(2) $$axy^3 - (x-a)^3y^2 + a^4x = 0.$$

Placing the equation on the triangle, we have at the origin $y^2 + ax = 0$. Fig. 8.

At an infinite distance,

near $(0, \infty)$, $xy + a^2 = 0$,

...... $(\infty, 0)$, $x^2y^2 = a^4 = 0$,

...... (∞, ∞), $ay - x^2 + 3ax - 3a^2 = 0$,

or $(x - \tfrac{3}{2}a)^2 = a(y - \tfrac{3}{4}a)$,

...... $(a, -a)$, $3a^2\eta - \xi^3 = 0$.

The curve is parallel to Ox where

$$a(y^3 + a^3) - 3(x-a)^2y^2 = 0,$$

$$\text{or } (y^3 + a^3)\{3x - (x-a)\} = 0;$$

$$\therefore y = -a, \quad x = a, \text{ and } x = \tfrac{1}{2}a.$$

Fig. 9.

(3) $a^3(y-x) - 2a^2(y^2 - x^2) + a(y^3 + x^3) - x^2y^2 = 0.$

The equation may be arranged

$$ay(y-a)^2 + ax(x^2 + 2ax - a^2) - x^2y^2 = 0,$$

$x = 0$, $y = 0$, or a, two values,

$y = a$, $x = 0$, or $\tfrac{1}{2}(\pm\sqrt{5} - 1)a = a$,

$x = 0$, $x = (\pm\sqrt{2} - 1)a$,

$x = (\pm\sqrt{2} - 1)a$, $y = 0$, and $(y-a)^2 - (3 \mp 2\sqrt{2})ay = 0$,

PLATE XIV. whence $y = \frac{8}{3}a$ and $\frac{1}{2}\frac{3}{8}a$ nearly for the upper sign, and $y = 7a$ nearly and a small value.

$$x = y = 2a, \text{ and } -x = y = a\sqrt[3]{2},$$

$$x = \pm a, \quad y^3 - 3ay^2 + a^2y + 2a^3 = 0,$$

whence $$y = 2a, \text{ or } \tfrac{1}{2}(\pm\sqrt{5} + 1)\,a.$$

$$\text{Near } (0, 0), \quad y - x + \frac{2x^3}{a^2} = 0,$$

$$\ldots\ldots\ (0, a), \quad \eta^2 - ax = 0,$$

$$\ldots\ldots\ (\alpha, a), \quad (3\alpha^2 + 2a\alpha - a^2)\,\xi - 2\alpha^2\eta = 0,$$

$$\text{or } (2\alpha + a)\,\xi - 2\alpha\eta = 0,$$

$$\alpha = \tfrac{5}{8}a, \qquad 9\xi = 5\eta,$$

$$\alpha = -\tfrac{13}{2}a, \quad 9\xi = 13\eta.$$

$$\text{Near } (\infty, \infty), \quad x^2 = a(y - 2a), \text{ and } y^2 = a(x + 2a).$$

The last parabolic asymptote meets the curve where

Fig. 10. $y^3 - 3ay^2 + a^2y + 2a^3 = 0$, the same equation as when $x = \pm a$.

(4) $$(y^2 - ax)(y^2 - bx) + x^4 = \pm c^4.$$

The equation may be written

$$\{y^2 - \tfrac{1}{2}(a + b)\,x\}^2 = \pm c^4 + \tfrac{1}{4}\{\tfrac{1}{2}(a - b)\}^4 - \{x^2 - \tfrac{1}{8}(a - b)^2\}^2,$$

$$y = 0, \quad x^2 = -\tfrac{1}{2}ab + \sqrt{(\tfrac{1}{4}a^2b^2 \pm c^4)}.$$

The curve is symmetrical with respect to Ox, and two values of y^2 are equal where

$$x^2 = \tfrac{1}{8}(a - b)^2 \pm \sqrt{\{\tfrac{1}{64}(a - b)^4 \pm c^4\}}.$$

i. Taking the positive sign, the compartments within which the curve lies are given by the two parabolas $y^2 = ax$, $y^2 = bx$, and the two lines $x = \pm c$, and the curve is parallel to Oy where it meets the parabola $y^2 = \frac{1}{2}(a + b)\,x$.

Fig. 11. The three parabolas are dotted in the figure.

ii. Taking the negative sign, $y^2 = ax$, $y^2 = bx$ are the

Fig. 12. only dividing lines, between which the curve must lie.

(5) $$y^2(x + y)^2(x - y) - 2a(x + y)\,x^2y - 4a^2x^3 = 0.$$

Near the origin the triangle shews that

$$y^5 + 4a^2x^3 = 0.$$

When x is infinite, y is finite,

$$y^2 - 2ay - 4a^2 = 0,$$

$$y = (1 \pm \sqrt{5})\, a.$$

Near (∞, ∞), $2(x+y)^2 + 2a(x+y) - 4a^2 = 0,$

$$\therefore\; x + y = -2a, \text{ or } a,$$

$$\text{and } x - y - a = 0;$$

$x + y + 2a = 0$ meets the curve at a finite distance where

$$4a^2y^2(x-y) + 4a^2x^2y - 4a^2x^3 = 0,$$

$$\text{or } yx(x+y) - y^3 - x^3 = 0;$$

$$\therefore\; x^2 - 2xy + y^2 = 0;$$

therefore this asymptote touches the curve at $(-a, -a)$; $x + y - a = 0$ meets it where

$$y^2(x-y) - 2x^2y - 4x^3 = 0,$$

or $-y^2 + 2xy - 4x^2 = 0$, roots impossible;

$x - y = a$ meets the curve where

$$y^2(x+y)^2 - 2y(x+y)x^2 - 4ax^3 = 0,$$

$$\text{or } y(x+y)(y^2 + xy - 2x^2) - 4ax^3 = 0,$$

$$\text{or } y(x+y)(y+2x) + 4x^3 = 0,$$

writing $y = zx$, $z^3 + 3z^2 + 2z + 4 = 0$,

if $z^2 = 2u$, $(u+1)(z+3) = 1$,

the hyperbola and parabola intersect where

$$z = -3 + \tfrac{2}{11} \text{ nearly.}$$

Fig. 13.

Note. The asymptotes parallel to Ox have each two points of the curve at a finite distance, and three points at an infinite distance, one of which is due to the point in which the parallel lines meet, so that in each case the curve lies on opposite sides of the asymptote at an infinite distance.

(6) $$x^5 = (x-y)^2(x+y)(x-2y),$$

$$x = 0,\; y = 0,$$

$$x = 1,\; y = 0,$$

$$\text{or } 2y^3 - 3y^2 - y + 3 = 0,$$

PLATE XIV.

which has only one real root a little less than unity,

$$x = \infty, \quad y = \infty.$$

Near (0, 0), $(x-y)^2 = -\frac{1}{2}x^3$, a ceratoid cusp,

or $x + y = \frac{1}{12}x^2$,

or $x - 2y = \frac{8}{3}x^2$.

Near (∞, ∞), $x^5 + 2y^4 = 0$.

The curve is parallel to Oy, where

$$3x^2 + xy - 8y^2 = 0;$$

whence it can be shown that

$$x = \tfrac{8}{2}y = -\tfrac{5}{81} \text{ nearly},$$

and $\quad x = -\frac{11}{6}y = \frac{24}{11}$ nearly.

It is parallel to Ox, where

Fig. 14. $\quad x = y$ and $x^2 = 2y^2$.

(7) $\quad x^6 + a^2y^4 + a^2x^3y + a^3xy^2 = 0.$

Fig. 15. Placing the equation on the triangle, the forms at the origin are given by

$$y^2 + ax = 0,$$

$$x^2 + ay = 0,$$

and $x^3 + a^2y = 0$,

where $x = y$, $x^3 + 2a^2x + a^3 = 0$,

and x lies between $-\frac{1}{3}a$ and $-\frac{1}{2}a$.

$$x = -y = -a.$$

Near $(-a, a)$, $\eta = 2\xi$,

Fig. 16. $y = -\frac{1}{3}x$ gives an idea of the size of the loop in xOy'.

(8) $\quad 4y^2(x+y-a)^3 = (x-y-b)^5.$

The equation may be put into a simpler form by making the lines $x+y-a=0$ and $x-y-b=0$ axes of coordinates, or writing

$$x + y - a = y'\sqrt{2},$$

and $x - y - b = x'\sqrt{2}$,

$$2y = (y' - x')\sqrt{2} + a - b;$$

$$\therefore\ y'^3(y' - x' + a)^2 = x'^5.$$

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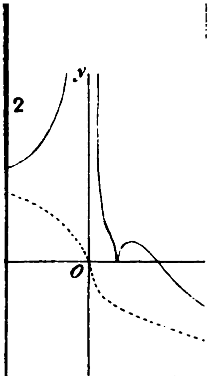

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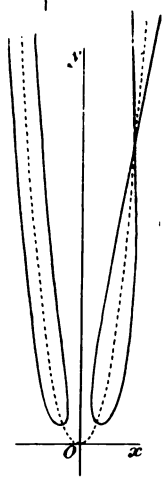

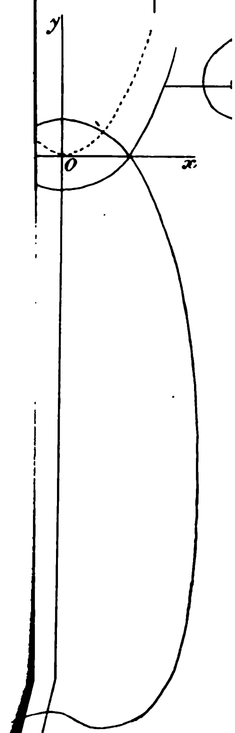

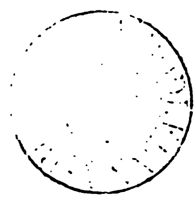

The lines parallel to the asymptotes through the new origin are $y'^3(y'-x')^2 = x'^5$. PLATE XIV.

If $x' = zy'$, $z^5 = (z-1)^2$, now the values of z in this equation are the abscissæ of the points of intersection of $x^3 = y^2$ and $y^2x^2 = (x-1)^2$, the latter of which is the two rectangular hyperbolas $(1 \pm y)\, x = 1$.

These only intersect in two points, whose common abscissa lies between $\frac{1}{2}$ and 1. Fig. 17.

If m be this value, let

$$z^5 - (z-1)^2 \equiv (z-m)\, \phi(z),$$

write $z = m + \zeta$, and equate coefficients of ζ;

$$\text{therefore } 5m^4 + 2(1-m) = \phi(m).$$

The equation of the asymptote is

$$x' - my' = \frac{2\alpha(1-m)}{\phi(m)}$$

$$= \frac{2\alpha(1-m)}{5m^4 + 2(1-m)},$$

$$\text{near the origin, } \alpha^2 y'^3 = x'^5,$$

$$\ldots\ldots (0, -\alpha), \quad \eta - x' = \left(-\frac{x'^5}{a^3}\right)^{\frac{1}{2}},$$

$$\ldots\ldots (\alpha, \alpha), \text{ if } x = \alpha + \xi, \; y = \alpha + \eta,$$

$$(\alpha + \eta)^3 (\alpha + \eta - \xi)^2 = (\alpha + \xi)^5,$$

$$\therefore\; 3\eta + 2(\eta - \xi) = 5\xi, \text{ or } 5\eta = 7\xi.$$

Fig. 18.

REPEATING CURVES.

205. I shall conclude this Chapter by shewing how to trace a large class of curves, whose equations involve trigonometrical functions of the coordinates in the place of the coordinates themselves. The loci of such curves, from the nature of a trigonometrical function, are made up of patterns continually repeated in every direction.

This symmetrical arrangement frequently gives very elegant figures, where the original curve has nothing to re-

PLATE XV. commend it from this point of view, trigonometrical functions acting in fact as a kind of kaleidoscope.

The following is the method I have adopted for tracing such curves.

206. If $f(x, y) = 0$ be the equation of any curve, and we write for y any trigonometrical function of x, and for x any trigonometrical function of y, a new equation will be formed of the kind which was spoken of; and it will be easy to see how any other functions can be treated, if we take the particular functions $\sin x$ and $\sin y$, and give constructions for the corresponding locus.

207. Take the equation of the curve to be

$$f(\sin y, \sin x) = 0 \quad \text{..................... i},$$

and construct the curve

$$f(\xi, \eta) = 0 \quad \text{..................... ii},$$

and the two curves of sines

$$\eta = \sin x \quad \text{..................... iii},$$

$$\text{and } \xi = \sin y \quad \text{..................... iv}.$$

Fig. 1. Let p be a point in the locus of (ii), which, by (iii) and (iv), must lie within a square whose sides AC, BC are parallel to the axes at the unit distance from them.

Let pr parallel to Ox meet (iii) in r, and ps parallel to Oy meet (iv) in s.

Draw sN, rM perpendicular to Oy, Ox, and let them intersect in P, P will be the point in the locus of (i) which corresponds to p.

$$\text{For } \eta = rM, \ \therefore\ OM = x \text{ by (iii)},$$
$$\xi = sN, \ \therefore\ ON = y \text{ by (iv)}.$$

Hence we have the following construction for the point P of (i) which corresponds to p of (ii).

Draw pr, ps, parallel to Ox, Oy respectively, meeting the curves (iii) and (iv) in r and s, and the point P will be at the angle of the rectangle Pr, Ps, which is opposite to p.

Observing that each of the lines Pr, Ps will meet their respective curves in an infinite number of such points as r, s, for each point p of (iii) there will be an infinite number of corresponding points, arranged in pairs symmetrically with respect to lines whose distances from the axes are odd multiples of $\frac{1}{2}\pi$.

208. In order to facilitate the tracing of any curve, such as (i), by constructing a sufficient number of points, the following properties should be noticed.

(1) When ps is a tangent to (ii), sP will be a tangent to (i); for, if p' be an adjacent point on PR, P' the corresponding point will be adjacent to P on sN. Fig. 1.

And similarly, if pr be a tangent at P, PM will be a tangent at P.

(2) Since ξ and η have no values beyond 1 and -1, the portion of the locus of (ii) which lies within the square, whose sides are given by $x = \pm 1$ and $y = \pm 1$, is the only portion which gives any part of the curve (i).

(3) If q be a point of (ii) which lies on the side of the square $x = 1$, and q' be an adjacent point, by considering where lines through q and q' parallel to Oy meet the curve (iv), it is seen that the points corresponding to q and q' are Q, and the pair of points Q', Q'', and the portion of (i) which corresponds to qq' is $Q'QQ''$ touching the line through Q parallel to Oy.

(4) From the first of the properties given above, it follows that, if the curve (ii) touch Ox at the origin, the curve (i) will touch Oy at the origin.

209. It is easy to see that similar constructions hold for the loci of equations in which $\sin mx$, $\tan mx$, &c., and $\sin ny$, $\tan ny$, &c., take the place of $\sin x$ and $\sin y$. The method of dealing with all these cases will be sufficiently shewn by a few illustrations.

PLATE XV.

(1) Take the curve whose equation is

$$\sin^3 y + \sin^3 x - 3a \sin x \sin y = 0 \text{ .... (a)},$$

derived from $$x^3 + y^3 - 3axy = 0 \text{ .... (b)},$$

which has been traced in pp. 98, 99.

i. Consider first the case in which a is small enough to allow the whole of the loop to lie within the square which limits the values of $\sin x$ and $\sin y$.

Fig. 2. Draw the curve (b), and suppose it the dotted curve apO in the figure; the curve being symmetrical with respect to the line $y = x$, it will be only necessary to construct the part of the curve (a) corresponding to the portion $abpOc$ of (b).

The principal points to be considered are:—a in the line $y = x$, b where the curve is parallel to Oy, O, and c where it is cut off by the limiting square. The corresponding points are A also in $y = x$, B where the curve (a) is parallel to Ox, O where the branch corresponding to Op touches Oy, and C where the curve (a) is parallel to Oy. The construction of Art. 207 is made in the figure for the points P, Q which correspond to two points p, q, in a line parallel to Ox and meeting the curve $y = \sin x$ in r, ps and qt, parallel to Oy, meet the curve $x = \sin y$ in s and t, and $Pspr$, $Qtqr$ are the rectangles spoken of in Art. 207. The shape of half the curve may then be drawn, and the other half by symmetry.

The manner in which the curve repeats itself is given in

Fig. 4. another figure.

ii. Consider next the case in which the loop of (b) intersects two sides of the limiting square.

Fig. 3. The points of (b) to be principally considered are:—a in $y = x$, b, d where it meets the side $x = 1$, O, and c in $x = -1$. The corresponding points are A, B and D when (a) is parallel to Oy, O where it touches Oy and C where it is parallel to Oy. The construction is given for two points p, q taken as before.

Fig. 4. The manner of repetition is given in another figure.

iii. The intermediate case, in which the loop touches the two sides of the square, is represented in a figure without any

Fig.

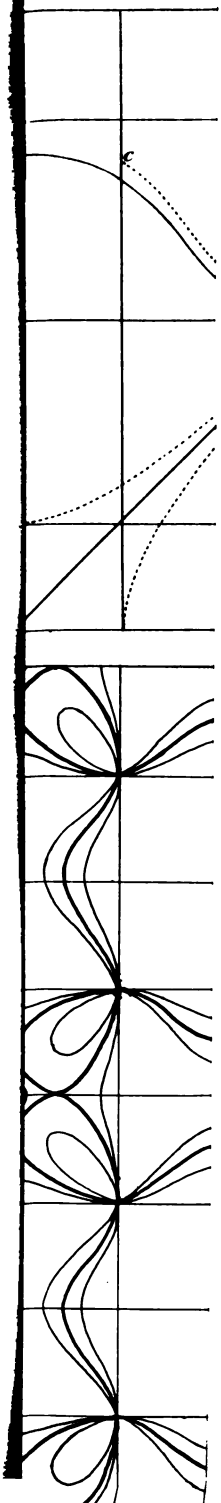

PLATE XV.

construction, by a darker line, the previous cases by lighter lines.

Note. The isolated portion reduces to a conjugate point D when the loop passes through the point (1, 1). Fig. 4.

210. The different effect obtained by introducing a tangent instead of one of the sines will be illustrated in the most striking manner by taking the same auxiliary equation.

Thus to trace the curve

$$\sin^3 x + \tan^3 y - 3a \sin x \tan y = 0.$$

The auxiliary equation is

$$\eta^3 + \xi^3 - 3a\eta\xi = 0.$$

In this case the curve will not be symmetrical with respect to the lines $y = x$, although the auxiliary equation is so; it will therefore be necessary to examine the positions of corresponding points for the whole of that part of the curve which is limited by the lines $y = \pm 1$.

The points to be first considered are:—a in $y = x$, b, c where the curve is parallel to the axes, and d, e on the limiting lines. The corresponding points are A, B, C, D, E. Fig. 5.

To shew how other points lie, p, q, r are points in a line parallel to Ox, and P, Q, R are the corresponding points lying in a line parallel to Oy, drawn through the point where pqr meets the curve of sines.

The figure which is derived from the case in which the loop cuts $y = 1$ in two points is given without naming the construction. Fig. 6.

PLATE XVI.

The manner of repetition is given in another figure, the dark line corresponding to the case of the loop touching $y = 1$. Fig. 1.

211. The figures are given without any construction for the two curves

$$\tan^3 x + \tan^3 y - 3a \tan x \tan y = 0,$$

$$\text{and } \sin^3 x + \sec^3 y - 3a \sin x \sec y = 0.$$

PLATE XVI.

Fig. 2. In the first curve, it is easily shewn that the tangent at $(\frac{1}{2}\pi, -\frac{1}{2}\pi)$ is $\xi + \eta = 0$.

Fig. 3. In the second, the curve is generated only from the small portions of the auxiliary curve which lie between $y = \pm 1$ and beyond $x = \pm 1$.

The groups marked α are derived from the portions between $y = 0$ and 1, and beyond $x = 1$.

Those marked β spring from the portions between $y = 0$ and 1, and beyond $x = -1$.

The conjugate points γ and that in the group β belong to the particular case for $a = 0$, and the auxiliary equation becomes $x + y = 0$, in which case there are only two points, $(1, -1)$ and $(-1, 1)$, which generate points in the curve.

In the group α of curves like Cartesian ovals, the conjugate points belong to the case of the loop of the auxiliary curve touching $x = 1$, corresponding to which is the dark oval in the group β.

The Lemniscate form is derived from the case in which the loop passes through $(1, 1)$.

In the group β, the oval between the dark lined oval and the conjugate point, with similar ovals, forms the entire curve, and this corresponds to a case in which the loop is entirely within $x = 1$ and $y = 1$, in which case the portion of the auxiliary curve joining the lines $x = -1$ and $y = 1$ is the generating portion.

212. By way of further illustration of the kaleidoscope property, I have given the patterns of the figures which correspond to three substitutions of trigonometrical functions in another equation

$$a(x+y)^2 = y(x^2 + y^2),$$

the dark lines corresponding to the case in which $a = \frac{1}{2}$, for which the auxiliary curve is parallel to Ox at the point $(1, 1)$.

Fig. 4. (1) $a(\sin y + \sin x)^2 = \sin x(\sin^2 x + \sin^2 y)$.

Fig. 5. (2) $a(\sin y + \tan x)^2 = \tan x(\sin^2 y + \tan^2 x)$.

Fig. 6. (3) $a(\tan y + \sin x)^2 = \sin x(\tan^2 y + \sin^2 x)$.

XI.

Trace the following curves:

(1) $y^4 + 2a^2xy - x^4 = a^4$.

(2) $a^3y = x^4 \pm a^2 (x - 2a) \sqrt{\{(x^2 - a^2)\}}$.

Shew which part of the curve belongs to the upper sign.

(3) $y^4 - m^4x^4 + a^2 (y^2 - xy - 2x^2) = 0$,

also the partial curves when $m = 1$.

(4) $y^2 = 4a\left(x + a \sin \frac{x}{a}\right)$.

(5) $x^5 = (x^2 - y^2)(x^2 - 4y^2)$.

(6) $x^5 = (x^2 - y^2)(2x^2 - 5xy + 2y^2)$.

(7) $x^6 + x^5y - 2bx^3y^2 + acy^2 = 0$.

(8) $x^6 = (a, b, c, d, e, f)(x, y)^5$,

giving the different forms corresponding to 5, 3, and 1 real roots of $(x, y)^5 = 0$.

(9) $x^6 + ax^4y + cx^2y^2 + dxy^3 + ey^4 = 0$.

(10) $x^7 + y^7 - 3a^2(x^5 + y^5) + 4a^3x^2y^2 = 0$.

(11) $x^4 + x^2y(3a + 2y) + y^3(a - y) = 0$.

(12) $(x^3 - y^3)^2 + 2a^2(x^4 - y^4) + a^4(y^2 - 4x^2) = 0$.

(13) $(y^2 - a^2)^3 - (5x + 6a)x^5 = 0$.

(14) $(x^2 + y^2 - 2ax)^2 = 2ax(y^2 - 3x^2 + 2ax)$.

(15) $y^2x^2 - 2ayx + b^2(y + 2b)^2 + 2(a^4 - b^4) = 0$.

(16) $(y^3 - x)^2 x^2 = (x^2 - 1)(x^2 - 4)$.

(17) $x^2y - y^2x = 2a(x - a)^2 - a(y - 2a)^2$.

(18) $(y - x^2)(y - x^3) - y^2x = 0$.

(19) $(x^2 - y^2)(x^4 + y^4) - (x^2 + y^2 - a^2)(x^2 + y^2 - 4a^2) = 0$.

(20) $x^3y^2 = (y - x^2)(y - 2x^2)(y - 3x^2)$.

PLATE XVI.

Trace the repeating curves:

(1) $m \sin x + n \sin y = 1$.

(2) $\tan y = m \sin x$.

(3) $\dfrac{\tan x}{\tan \alpha} + \dfrac{\tan y}{\tan \beta} = 1$.

(4) $\dfrac{\sin^2 x}{\sin^2 \alpha} + \dfrac{\sin^2 y}{\sin^2 \beta} = 1$.

(5) $\tan x \tan y = m$.

(6) $\sin x \sin y = m$.

(7) $\sin^2 y = n \sin x$.

(8) $\cos^3 x + \sin^3 y - 3a \cos x \sin y = 0$.

CHAPTER XII.

INVERSE PROCESS. DETERMINATION OF THE EQUATION OF A GIVEN CURVE.

PLATE XVII.

213. I PROPOSE in this last chapter to say a few words on methods of discovering equations which will represent the general form of a traced curve, at all events when it is known that the curve is capable of representation by means of an algebraical equation.

Of course there are difficulties which are unavoidable in the way of the exact determination of all the coefficients of the terms of an equation, some of which it will be well to point out, so that too much may not be expected from the treatment of this kind of problem.

214. If a curve could be supposed to be so accurately drawn that it would stand the test of measurement in every part, we could theoretically obtain equations for determining all the constants which would appear in the general equation of any degree which we might judge the curve to require. But in the case of curves which run off to infinity, the observations which I have made in Art. 25, are sufficient to explain the impossibility of such accuracy in some of the most important parts of the curve. Again, it is impossible, except by some sort of convention as to drawing, to distinguish the degrees of closeness of contact of such parabolic forms as are given by $y^2 = ax$, $y^4 = ax^3$, $y^4 = a^3x$, and the misappreciation of the order of contact would affect the degree of the equation to be tried.

Such difficulties might be met by a verbal statement of the nature of the asymptotes and the forms at particular

PLATE XVII.

important points, at the same time that the curve is given in its general features.

If such statement be not made we must make the best approximation to which our experience in the direct treatment of curves may guide us.

215. The impossibility of the representation of a given curve by an algebraical equation would manifest itself in a variety of ways. A breach of continuity, for example, might exhibit itself by the curve passing off to an infinite distance by one branch without returning by another; for the continuity involved in an algebraical equation requires that, when a curve disappears along one branch of a rectilinear or parabolic asymptote, it should reappear either at the same or the opposite end, just as much as that the curve should not stop short at a finite distance; this could only be represented by the introduction of discontinuous functions.

216. In spite of the uncertainty which the difficulties mentioned above introduce, I think this exercise in the highest degree profitable, as an illustration of the manner in which, in physical subjects, difficulties arising from uncertain data and imperfect measurements are met and theories formed, which, as science advances, become nearer and nearer representations of the results given by experiments.

This must be my excuse for making this attempt, however imperfect, to shew how to deal with a problem which is in its nature not very precise.

217. The most obvious step towards this inverse process is to find as nearly as possible the degree of the curve. A lower limit to the degree is given at once by finding the greatest number n of points in which a straight line, drawn in the most favourable position, can be made to intersect the curve; since, in any position, there may be imaginary points of intersection, the degree of the equation may be $n + 2m$ where m is any positive integer.

PLATE XVII.

For example, a curve whose equation is $x^4 + y^4 = c^4$ can only be cut by a straight line in two distinct points. Such a curve might be distinguished from a circle by its flatness where it meets the axes, but it would be hard to distinguish it from the curve $x^8 + y^8 = c^8$.

However, there will generally be some position of an intersecting line which will give as many real points of intersection as are exactly equal to the degree of the curve; at all events, it would be advisable to make the first attempt with such a degree.

218. The next necessary step is to select favourable positions for the coordinate axes to which it is intended to refer the curve. In this choice we are guided by many considerations, among which are symmetry, relations to the directions of asymptotes, contact of the axes with branches of multiple points, or, if we observe that there is one direction parallel to which straight lines cut the curve in a smaller number of points than in any other direction, it may be advisable to take one of the axes, as that of x, in that direction, since the highest power of x which would then appear in the equation would be less than for any other direction, and there would be a consequent diminution of the number of terms of the general equation which would have to be introduced.

The advantage of having the origin at a multiple point or cusp is obvious.

219. The terms involving the highest powers of one or both coordinates can be found from the first approximations to the asymptotic branches, if there be any, and those of the lowest powers from the forms near the origin. We must then adjust the intermediate terms so as to satisfy the conditions of magnitude and other peculiarities presented by the curve.

If, when a sufficient number of conditions have been taken into consideration to determine these intermediate terms, it should be found that other conditions are not satisfied, this

PLATE XVII.

would shew that an equation of a higher degree ought to have been tried, or that some error of magnitude has been made.

When the curve has not many special points, these tentative methods will be generally sufficient for the purpose.

In more complicated curves other processes will be required, but before proceeding to these I shall give a few examples of what I have been observing.

(1) Take the curve Fig. 28, Pl. III.

The curve is probably of the fourth degree.

If the axes be chosen as in the figure, the symmetry shews that no odd powers of y enter the equation.

The shape at the origin would be represented by two parabolas $(y^2 - ax)(y^2 - bx) = 0$.

The multiple point on the axis of x would be represented by $y^2 = m^2 (x - c)^2$.

The equations given by the highest powers equated to zero must give imaginary results, since there are no asymptotes.

To satisfy these conditions the equation may be of the form

$$y^4 - \alpha cxy^2 + \beta x^2y^2 + \gamma^2x^2 (x - c)^2 = 0.$$

The conditions of magnitude to be satisfied appear to be when $x = c$, $y = 2c$, therefore $\alpha - \beta = 4$; when $x = 2c$, $y = c$ or $2c$, therefore $2\alpha - 4\beta = 5$, and $\gamma^2 = 1$; whence $2\alpha = 11$, $\beta = \frac{3}{2}$, and these values give, near $(c, 0)$, $4y^2 = (x - c)^2$, and near the origin $y^4 - \frac{11}{2} cxy^2 + c^2x^2 = 0$.

This does not coincide with the equation in the text, but sufficiently represents the general features, the discrepancy arising from a misappreciation of the magnitudes involved. With good drawing and accurate movements as near an approximation as we like could be made.

(2) Take the curve Fig. 28, Pl. IV.

This curve appears to be of the fifth degree.

Choosing the axes as in the figure, the form at the origin may be represented by $x^3 = ma^2y$.

For any value of x there are only two values of y.

The equations of the asymptotes can be represented by $x=a$, $-a$, and $2a$, and $y=\pm a$.

The equation of the curve may therefore be written $(x^2-a^2)(x-2a)y^2-a^2x^3+ma^4y=0$.

When $x=0$ $y=2a$, therefore $m=4$; this solution gives $(2a, 2a)$ as a point in the curve, and it is easily seen that the curve lies on the proper side of the asymptotes.

The equation deduced is the same as that given in the text.

(3) Take Fig. 21, Pl. IV.

The curve may be of the fourth degree; near the origin $y^3=Ax$, apparently; when $x=0$, $y=-2a$; the axis of x is an asymptote with a second approximation $y^2=-\dfrac{ra^3}{x}$; the two cross asymptotes are $(y+a)^2-x^2=0$.

The equation to be tried is therefore

$$y^2(y^2-x^2)+2ay^3-ra^3x+ma^2xy=0.$$

The asymptote $y+x+a=0$ meets the curve where

$$-ay^2(y-x)+2ay^3-ra^3x+ma^2xy=0,$$

$$\text{or } ay^2(y+x)+a^2x(my-ra)=0;$$

$$\text{therefore } y^2+(my-ra)(y+a)=0,$$

$$\text{or } (m+1)y^2-(r-m)ay-ra^2=0,$$

now both roots are negative, and numerically less than a and greater than $2a$, these conditions are satisfied by $m=-5$, $r=9$.

The equation thus found, viz.

$$y^2(y^2-x^2)+2ay^3-9a^3x-5a^2xy=0,$$

differs materially from that in the text, the fact being that there is an error in the drawing of the curve, the asymptote of which should have been cut on the positive side of Ox, and the curve should have been above to the left and below to the right, the error having been just now discovered in the course of this inverse process.

PLATE XVII.

(4) Take Fig. 17, Pl. III.

The curve may be of the fourth degree; the tangents at the origin, $y \pm x = 0$ and $y - 2x = 0$; the curvilinear asymptote y^3 or $y^4 = Ax$ or $y^4 = Bx^3$.

If, when $x = 0$, $y = a$, the resulting equation would be

$$a(y^2 - x^2)(y - 2x) - y^4 = 0;$$

since no other term would be admissible.

This is the equation given in the text.

METHOD BY THE TRIANGLE.

220. In more difficult cases it will be advisable to save trouble by making use of the properties of the analytical triangle, by means of which, when the terms which give the directions of infinite branches and branches of multiple points and cusps at the origin have been determined, it is seen exactly what intermediate terms may be introduced without affecting these directions.

221. The arrangement of terms corresponding to infinite branches and multiple points is made easy by considering the property proved in Art. 150, that all parallel lines containing circles which indicate terms of an equation give, when such are equated to zero, the same relation between x and y as far as degree is concerned; so that, if there be, for example, a point of inflexion at the origin, the branch touching one of the axes, an equation giving such a form would be $y^r = Ax^s$, where r and s are different odd numbers, and the same relation would be given for all parallel lines.

The polygon must, therefore, have one of its lower sides parallel to the line $\dfrac{x}{r} + \dfrac{y}{s} = 1$.

A similar argument holds for asymptotes with respect to the upper sides of the polygon.

If there be a rectilinear asymptote, one of the sides of the polygon must be parallel to the hypothenuse of the triangle.

If the asymptote pass through the origin there is generally

no circle at those angles which the side first encounters, when moved parallel to itself towards the right angle.

If the curve cut one of the axes, we must take care that circles are placed on the corresponding side of the triangle, sufficient to give as many solutions as there are points of intersection.

222. Having thus selected convenient axes, and determined the direction of lines corresponding to tangents to the branches through the origin, as well as to equations representing first approximations to the asymptotic branches, a polygon must be drawn whose sides shall be parallel to these and contain no re-entering angles, the upper sides corresponding to the asymptotic branches, and, if the axes of coordinates intersect the curve, two of the sides being coincident with the sides of the triangle.

It will be seen that the sides of the polygon may often be made to cover less ground by the introduction of additional sides, taking care that the corresponding equations give imaginary solutions.

223. When the polygon is completed, circles indicating terms must be placed in the interior, so as to satisfy the other conditions of magnitude and position presented by the curve. We must then trust to our ingenuity, or, if we can rely on the correctness of the drawing, to our measurements, to determine suitable coefficients for the terms.

224. The advantage gained by this use of the triangle may be seen at once by trying to reproduce the equation of the curve given Pl. VI. Figs. 25 and 26, which may be of the seventh degree, the general equation of which would contain twenty-eight terms.

The polygon, whose sides produce the form at the origin and at infinity, may belong to four terms which, on trial, will be found to be sufficient. If they had not been sufficient to represent the form of the curve in other respects, any of the six terms between x^7 and y^7 might have been inserted,

PLATE XVII. having taken care that there was no real solution but $x+y=0$, one term x^6y might have been introduced between x^7 and x^8y^2 which would have had the effect of making the latera recta of the parabolic forms different, only three terms more could have appeared, viz. x^2y^4, x^3y^3, x^4y^2. Even if every possible additional term had been introduced, the reduction of terms to be tried would have been very great.

225. Both in the direct and inverse process Des Cartes' rule of signs is invaluable as a test of correctness; thus, in the curve just discussed, when x has some positive value there ought to be two or four changes of sign in the equation y, and when y changes sign five, or three changes. This enables us often to say at once whether more terms must be taken in on trial.

226. I shall give a small number of examples of the manner in which, when the directions of the sides of the polygon are determined, they may be placed end to end, without attempting in every case to complete the equation in other respects.

Fig. 1. 227. In the curve represented in the figure, if we make the point A the origin, and the tangent at the point of inflexion the axis of y, the forms at the origin could be re-
Fig. 2. presented by $y^3=-a^2x$ and $x^3=by^2$, giving directions ab, cd of two lower sides of the polygon.

When $y=0$, $x+c=0$, so that one side of the polygon is on a side of the triangle.

$x^3=d^2y$ will apparently represent the infinite branches, and an upper side must be parallel to ed.

The polygon $\alpha\beta\gamma\delta\varepsilon$ satisfies these conditions, with one side $\gamma\delta$ which we must take care to make correspond with an equation having imaginary solutions.

We thus obtain an equation of the lowest degree which will satisfy the limiting conditions.

Again, since the form at B would be given by $y^3=e^2(x+c)$, we must take care that the coefficients of y and y^2 vanish

near the point $(-c, 0)$, and the only terms admissible, which involve y and y^2, must be of the form $A(x+c)x^3y$ and $(Bx^3+Cx^2+Dx)(x+c)y^2$. PLATE XVII.

The simplest equation to be tried is therefore

$$y^5 - d^2x^3y^4 - \beta x^3y^3 + b\gamma x(x+c)y^2 - \gamma(x+c)x^4 = 0.$$

Applying the test of Des Cartes' rule,

$$\text{if } 1 > x > 0,$$

$$\begin{array}{lllllll} y+ & \text{gives} & + & - & - & + & - \\ ..- & ...... & - & - & + & + & - \end{array}$$

consistent with one +, and two or no − value of y;

$$\text{if } 0 > x > -c,$$

$$\begin{array}{lllllll} y+ & \text{gives} & + & + & + & - & - \\ ..- & ...... & - & + & - & - & - \end{array}$$

$$\text{if } -c > x,$$

$$\begin{array}{lllllll} y+ & \text{gives} & + & + & + & + & + \\ ..- & ...... & - & + & - & + & + \end{array}$$

so also for the values of x corresponding to any value of y.

This form of the equation will therefore probably represent the curve with tolerable accuracy.

228. The curve represented in the figure may be of the sixth degree; the shapes near the origin may be $x+y=0$, and $y^2=Ax$; when $y=0$, $x=2a$; the asymptotes to the first approximation are $(x-y)^2=0$, $y=a$, and $xy^2=B$. Fig. 3.

The polygon producing these is given in the next figure. Fig. 4.

Towards the determination of what terms ought to be taken in within the polygon, we observe that, the curve passing on both sides of one end of the asymptote $\delta\epsilon$, the next approximation must be $(x-y)^2=\frac{c^3}{x}$, which would require the introduction of terms involving x^2y or xy^2. Again, the equation corresponding to $\epsilon\zeta$ must give only one real solution, and we can try $y^3-a^3=0$, and since the curve

PLATE XVII. lies below at both ends, the coefficient of x^2 must vanish and that of x be positive.

Writing the equation with some undetermined coefficients

$$xy^3(x-y)^2 - a^3x^3 - b^3y^3 + Ax^2y + Bxy^2 + 2a^4x(x+y) = 0.$$

If $y = a$, coefficient of $x^2 = -2a^4 + Aa + 2a^4$, $\therefore A = 0$;

$$\ldots\ldots\ldots\ldots\ldots \quad x = a^5 + Ba^2 + 2a^5$$

$$= (r+3)\,a^5,\ r+3 \text{ positive};$$

near (∞, ∞), $x(x-y)^2 = a^3 + b^3 - ra^3 = s - r + 1$, $s > r - 1$, and the equation is

$$xy^3(x-y)^2 - a^3x^3 - sa^3y^3 - ra^3xy^2 + 2a^4x(x+y) = 0.$$

The next approximation to $x + y = 0$ is

$$a(x+y) = \tfrac{1}{2}(1 - s + r)\,x^2,\ s > r + 1;$$

$$\text{when } x = y,\ (1 + s + r)\,x = 4a,$$

$$\text{when } x = -y,\ 4x^6 = (s - r - 1)\,a^3x^3.$$

The conditions of magnitude seem to be satisfied by

$$1 + s + r = 2 \text{ and } s - r - 1 = 4,$$

$$\text{which give } s = 3 \text{ and } r = -2.$$

The final equation is

$$xy^3(x-y)^2 - a^3x^3 - 3a^3y^3 + 2a^3xy^2 + 2a^4x(x+y) = 0,$$

which satisfies the conditions fairly, for

$$\text{near } (2a, 0),\ 4a^5\xi = 2a^4y,$$

$$\text{where } y = a,\ a^5x - 3a^6 + 2a^5x + 2a^5x = 0,$$

$$\therefore x = \tfrac{3}{5}a;$$

$$y = -2x \text{ gives } 72x^4 - 31a^3x + 2a^4 = 0,$$

which has no negative root, one small positive value, and one nearly $\frac{3}{4}a$.

The rule of signs agrees with both x and y when y and x are given.

Fig. 5. 229. The curve in the figure appears to be of the fourth degree.

The approximate form near the origin may be given

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by $(x-y)^2 = Ax^3$; near $(0, \infty)$ by $xy = a^2$; near (∞, a) by $y - a = -\frac{b^3}{x^2}$, also $y = 0$, $x = -a$.

The polygon therefore would be as in the next figure. Fig. 6.

The equation, if of the fourth degree, must be

$$a^2(x-y)^2 - (y-a)x^3 - Ayx^2 - Bxy^2 - Cxy^3 - Dx^2y^2 = 0.$$

$$\text{Near the origin, } a(x-y)^2 = (A+B-a)x^3,$$

$$\therefore\ A + B - a \text{ is positive;}$$

$$\text{when } y = a, \text{ coefficient of } x^2 = a^2 - Aa - Da^2 = 0,$$

and that of x is negative,

$$\therefore\ -2a^2 - Ba^2 - Ca^2 \text{ is negative.}$$

The roots of $yx^3 + Cxy^3 + Dx^2y^2 = 0$ must be imaginary, or $4C > D^2$.

Observing the apparent magnitudes, when $y = a$, $x = \frac{1}{3}a$, which gives $B + Ca = a$.

$$x = y = -a \text{ gives } 2a + a(C+D) = A + B,$$

$$\text{and } A + Da = a;\ \therefore\ C + D = 0.$$

If $D = -\frac{1}{4}$ we obtain the values

$$C = -\tfrac{1}{4},\quad A = \tfrac{5}{4}a,\quad B = \tfrac{3}{4}a,$$

and the equation becomes

$$4a^2(x-y)^2 - 4(y-a)x^3 - 5ayx^2 - 3axy^2 - xy^3 + x^2y^2 = 0.$$

METHOD BY PARTIAL CURVES.

230. It is frequently possible to discover, from the form of the curve, simple partial curves, from which the given curve may be supposed to have degenerated. If we then form a single equation which will represent these partial curves, we may obtain at once terms which will form the required equation, by a proper alteration of the coefficients, or by the introduction of such terms as will prevent the equation from being split into simpler equations.

231. A simple case of the application of this method is to the curve represented in Fig. 15, Pl. II., which may be

PLATE XVII. Fig. 7. supposed to be degenerated from a circle and two straight lines, as in the figure.

The equation representing the partial curves is

$$(x^2 - y^2)(x^2 + y^2 - c^2) = 0,$$

from which we obtain an equation

$$x^4 - y^4 - a^2x^2 - b^2y^2 = 0,$$

which, if a be a little greater than b, would give the required curve.

Fig. 8. 232. The curve represented in the figure can be conceived to have degenerated from the three dotted circles, whose equation is

$$\{(x^2 + y^2)^2 - a^2x^2\}(x^2 + y^2 - 4a^2) = 0,$$

and, considering the compartments within which the curve lies, the equation may be written

$$\{(x^2 + y^2)^2 - a^2x^2\}(x^2 + y^2 - 4a^2) + \alpha a^6 = 0,$$

which will produce the given curve if α be such as to give, when $x = 0$, two positive values of y^2 a little less than $4a^2$ and a^2, since one value of y^2 will be negative; and the values of x^2, when $y = 0$, will be two impossible and one negative, the solutions of the limiting equation being impossible.

If there were a conjugate point at the origin we might write $a^4(\alpha x^2 + \beta y^2)$ for αa^6.

Fig. 9. 233. The curve in the figure may be obtained from the partial curves whose equation is

$$(x^2 + y^2 - a^2)(x^2 - b^2)(y + c) = 0.$$

For assuming the equation

$$(x^2 + y^2 - a^2)(x^2 - b^2)(y + c) + ca^2b^2\gamma^2 = 0,$$

where γ is small compared with a and b,

$$\text{if } y = 0; \quad (x^2 - a^2)(x^2 - b^2) + a^2b^2\gamma^2 = 0;$$

$$\therefore\ x^2 = a^2 - \frac{a^2b^2\gamma^2}{a^2 - b^2} \text{ and } b^2 + \frac{a^2b^2\gamma^2}{a^2 - b^2} \text{ nearly,}$$

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which accounts for the two side loops.

If $x=0$, $(y^2-a^2)(y+c)=ca^2\gamma^2$,

$$\therefore\ y=\pm\left\{a+\frac{ca^4\gamma^2}{2(c\pm a)}\right\}\ \text{and}\ -c+\frac{ca^2\gamma^2}{c^2-a^2},$$

which accounts for the lower loop.

The side loops degenerate into conjugate points if

$$(a^2+b^2)^2=4a^2b^2(\gamma^2+1),\ \text{ or }\ \gamma=\frac{a^2-b^2}{2ab}.$$

If $-\gamma^2$ be written for γ^2 we obtain the light curve.

234. The following example will shew how the use of compartments may be combined with that of partial curves to obtain the equation of some curves.

235. Fig. 10 and Fig. 11 represent curves which can plainly be derived from an ellipse and straight line as given by the dotted lines.

The equation of the Fig. 10 is of the form Fig. 10.

$$(ax^2+by^2-1)(y-c)+\alpha^2=0,$$

where α is a small quantity compared with those employed in the partial equations; for in the case of the oval ax^2+by^2-1 is negative, and $y-c$ positive, and *vice versâ* for the remaining portion of the curve.

To find the equation of the Fig. 11 we have to contrive that the partial curves shall be cut by the given curve in the three points A, B, C; as these points are placed in the figure they would lie in a parabola, and the equation of the curve would be of the form Fig. 11.

$$(ax^2+by^2-1)(y-c)+\alpha^2(y^2-bx)=0.$$

If B had been in the line AC, the equation would have been

$$(ax^2+by^2-1)(y-c)-\alpha^2(x-d)=0.$$

236. The curves given in Fig. 12 and Fig. 13, are clearly derivable from a circle, and three lines forming an inscribed triangle.

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The equation of the Fig. 12 is of the form

Fig. 12.

$$(x^2+y^2-2cx)(y^2-m^2x^2)\{(1+m^2)x-2c\}+\alpha^2=0;$$

Fig. 13. and since the curve in the Fig. 13 crosses the partial lines in A, B, C, D, E which lie in a line, the equation is of the form

$$(x^2+y^2-2cx)(y^2-m^2x^2)\{(1+m^2)x-2c\}-\alpha^2(ax+by-1)=0.$$

XII.

As examples the equations of the Figs. 14—20 may be found, separate curves being drawn in Fig. 19 with hard and dotted lines.

For general examples of this chapter, equations may be investigated to represent the curves 1, 2, 6, 9, 13, 15—18, 20, 21, 25, 28 traced in Pl. V.

THE END.

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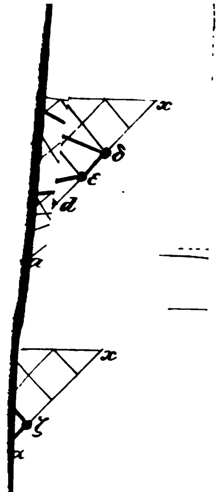

Fig. 10